Chas. D. Yardley

THE ELEMENTS OF MECHANICS.

By the same Author,

A FIRST BOOK OF NATURAL PHILOSOPHY; or, an Introduction to the Study of Statics, Dynamics, Hydrostatics, Optics, and Acoustics, with numerous Examples. Fourteenth Thousand. Crown 8vo, 3s. 6d. Cloth.

MATHEMATICAL EXAMPLES: a graduated series of Examples in Arithmetic, Algebra, Logarithms, Trigonometry, and Mechanics. Crown 8vo, 8s. 6d. Cloth.

THE ELEMENTS OF MECHANICS;

INCLUDING

HYDROSTATICS:

WITH

Numerous Examples.

BY

SAMUEL NEWTH, M.A., F.R.A.S.,

FELLOW OF UNIVERSITY COLLEGE, LONDON;
PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY IN NEW COLLEGE, LONDON;
AND ASSISTANT MATHEMATICAL EXAMINER IN THE UNIVERSITY OF LONDON.

FIFTH EDITION.

LONDON:
JAMES WALTON,
BOOKSELLER AND PUBLISHER TO UNIVERSITY COLLEGE,
137, GOWER STREET.
1869.

PLYMOUTH:
PRINTED BY WM. BRENDON AND SON,
GEORGE STREET.

PREFACE TO FOURTH EDITION.

IT is the design of the following Work to supply a manual, by the aid of which a Student may intelligently prosecute the study of Natural Philosophy in the earlier stages of his Mathematical course.

The Work is divided into two parts. The First Part embraces the more elementary portions of Mechanics, and can be read by a Student who has advanced as far as the earlier propositions of the Sixth Book of Euclid, and the Solution of Quadratic Equations. Those Sections in which Trigonometrical expressions are employed, are so arranged that they may be omitted on a first reading. In the Second Part, the subjects are carried on to the point at which a knowledge of the Differential Calculus becomes necessary.

Several new Sections and additional Examples have been introduced in the present Edition. The changes thus made in the Work will, it is hoped, increase its utility, as well

for the purposes of general Education, as for those who are preparing for University Examinations.

The First Part contains all the subjects in Mechanics and Hydrostatics required for the B.A. and B.Sc. Examinations in the University of London. In order to meet some recent changes made in the subjects of Examination, a Chapter has been added on the simpler cases of motion about centres of force. These, as generally treated, demand a more extended knowledge of Algebraical Geometry than is explicitly required from Candidates for these degrees. This I have endeavoured as far as possible to avoid, and in an Appendix have deduced the few expressions to which I have found it necessary to appeal.

SAMUEL NEWTH.

NEW COLLEGE, LONDON,
June, 1864.

CONTENTS.

Part I.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

CHAPTER XIV.

CHAPTER XV.

CHAPTER XVI.

CHAPTER XVII.

Part II.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

"Force is an action between two bodies either causing or tending to cause change in their relative rest or motion."
Rankine's Applied Mechanics. Art. 12.

Part I.

STATICS.

CHAPTER I.

DEFINITIONS AND AXIOMS.

1. Whatever is capable of producing motion in a body, or any *change* in the motion of a body, is termed *force.*

In other words, force is the name we employ to express that unknown cause which, under any circumstances, can produce a change in the state, whether of rest or motion, of any material body.

Whatever causes a change in the motion of a body must be regarded as of like nature with that which produces motion, and hence the same term (force) is applied to both, even although there are some forces which, while they are able to change the motion of a body, can never produce it. Such, for example, are friction and resistances of all kinds. Forces of this nature can, it is evident, never act alone, for some other force must be present in order to produce the motion which they change; and hence, if but one force act upon a body, it must be one capable of *producing* motion.

2. If one force only act upon a body, motion must necessarily follow; but when two or more forces act upon the same body,

their united effect *may* be such that no motion ensues. Whenever this is the case, the forces are said to be *in equilibrium.*

3. That branch of mechanics which investigates the relations which exist between forces in equilibrium is termed Statics; and that which investigates the effects of forces not in equilibrium, but producing motion, is termed Dynamics.

4. Whenever motion is prevented by muscular effort, as for instance, when a weight is held in the hand and so prevented from falling to the ground, or when an elastic cord is stretched and its rebound prevented, a sensation is produced which we call *pressure.* But, just as the same word is used in several other cases both for the sensation and its cause—*e.g.* sound, smell, taste —so is the term pressure applied also to the force whose resulting motion has been thus prevented. In this sense, then, pressure is force considered as the cause of the sensation which is felt when motion is prevented by muscular effort. It is with this meaning that the term is commonly employed in mechanics, although extended to all forces when the motion they are capable of producing is in any way prevented, whether it be by muscular action or not. Thus, whether a weight be held in the hand or rest upon a table, in either case it is said to exert a pressure, and a pressure of the same amount.

The commonest case of pressure is weight, and this supplies the most convenient standard of reference by which to compare different pressures. By means of weight, other pressures may in most cases be easily measured. Thus, when an elastic cord is held stretched by the hand, the pressure it exerts may be compared with others, by determining what weight will keep the string stretched to the same degree.

Since all questions considered in Statics refer to forces in equilibrium, all statical forces may be denominated pressures, and consequently may be measured by weight.

5. Forces may differ from each other, not only in magnitude,

but also in direction, and hence may be conveniently represented by straight lines; the direction of the lines representing the direction of the forces, and the lengths of the lines, each being measured by the same scale, the magnitudes of the forces.

Two forces will be called *concurrent*, if they both act towards or from the same point. In the case of parallel forces, and of forces acting along the same straight line, this point is supposed to lie at an infinite distance.

6. Force, when transmitted by means of a cord, is sometimes spoken of as the *tension* in the cord.

7. The single force, which represents the combined effect of several forces, is termed their *resultant;* relatively to the resultant, these several forces are termed *components* or *component forces. Composition of forces* takes place when two or more forces are replaced by a single force equivalent to them, that is, when the resultant is substituted for its components. *Resolution of forces* takes place when a single force is replaced by two or more forces equivalent to it, that is, when the components are substituted for their resultant.

8. A *rigid* body is one the relative position of whose particles is supposed to be invariable.

9. AXIOMS. i. If two non-concurrent forces acting in the same straight line are in equilibrium, they are equal. $+x - y = 0$. $x = y$

ii. If two non-concurrent forces acting in the same straight line are equal, they are in equilibrium. $+y - y = 0$

iii. The resultant of two forces acting in the same straight line is their sum when the forces are concurrent, and their difference when non-concurrent. $x \pm y$

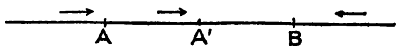
iv. The resultant of two concurrent forces acting along different lines falls between them.

v. The effect of any forces acting upon a rigid body is unaltered

by the introduction or the removal of any number of forces that are mutually in equilibrium. $a+b = a+b+q-q$

10. Transmissibility of Force. *A force acting upon a rigid body may be supposed to act at any point, within the body, in the line of its direction.*

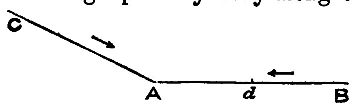
Thus, if a force P act upon a rigid body at A, and along the line AB in the direction of the arrow, we may suppose P to be applied at any other point A′ in the line AB within the body.



For at A′ and B introduce two non-concurrent forces, each of the same magnitude as P. The forces at A and B are (Ax. ii.) in equilibrium, and may, therefore, by Ax. v., be removed. There will then remain only a force P at A′ of the same magnitude as the original force, and acting in the same direction.

11. *If two forces be in equilibrium their directions must be exactly opposite.*

For, if possible, let two forces acting upon any body along the lines CA, BA, be in equilibrium; since there is equilibrium, there will not less be equilibrium, if any point be supposed fixed.



Let d in the line AB be such a point, the force acting along AB will be met by the re-action of the fixed point d. The force CA will then remain alone, and will turn the body round the fixed point d, and therefore there will not be equilibrium. Consequently, if two forces be in equilibrium, their directions must be exactly opposed, and hence, by Axiom i., they are also equal in magnitude.

12. If any number of forces be in equilibrium, any one force must be equal in magnitude and opposite in direction to the resultant of the remaining forces. For if, instead of the remaining forces, we substitute their resultant, we shall have but two forces; namely, this resultant and the force in question, and these are in equilibrium, and therefore are equal and opposite.

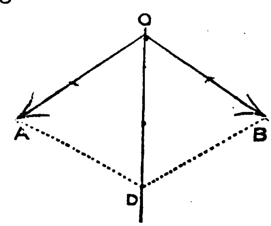
CHAPTER II.

ON THE COMPOSITION AND RESOLUTION OF FORCES ACTING UPON A POINT.

13. Parallelogram of Forces. *If two concurrent forces acting upon a point are represented in magnitude and direction by the two* adjacent *sides of a parallelogram, then will their resultant also be represented in magnitude and direction by the diagonal drawn through the given point.*

First. If two equal forces act upon a point, the *direction* of the resultant is that of the diagonal, [illegible]

Let AO, BO represent the magnitude and direction of two equal forces acting upon the point O, the diagonal OD must be in the direction of the resultant: for since OADB is an equilateral parallelogram, the diagonal OD bisects the angle AOB; and since the forces acting upon O are equal, there can be no reason why the resultant should pass nearer to the one than to the other; it must therefore pass at an equal distance from both: consequently it must pass along OD.

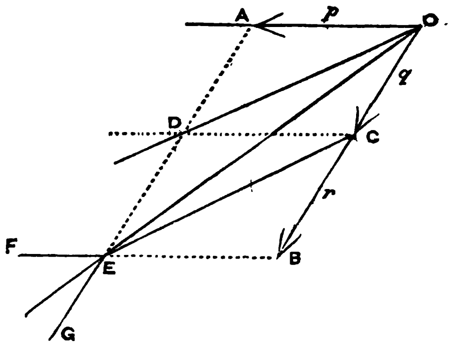


Secondly. If the direction of the resultant be that of the diagonal, in the case of forces whose magnitudes are p and q, and of forces whose magnitudes are p and r, it is so also in the case of forces whose magnitudes are p and $q+r$.

Let AO, BO represent two forces of the magnitudes p and $q+r$ respectively, acting upon the point O, and suppose them to be kept

in equilibrium by some unknown force acting upon O. Let CO $= q$, then BC $= r$.

Complete the parallelograms AC and DB. Draw the diagonals OD, OE, and CE. For the forces p and q in AO and CO substitute their resultant, which, by the hypothesis, is a force acting along OD. Suppose this force to act at D (Art. 10), then resolving it, we have a force p acting along the line DC, and a force q acting along the line DE. Let these two forces act at C and E respectively.



Again, for the forces p in CD and r in CB, substitute their resultant, which, by hypothesis, is a force acting along CE. Let this force act at E, and resolve it. We shall have then at E a force p acting in EF, and two forces q and r acting in EG. We have consequently, without disturbing the equilibrium, removed the original forces p and $q + r$ from O to E. The resultant, therefore, of the given forces must pass through E. It must also pass through O, and therefore must act along the diagonal OE.

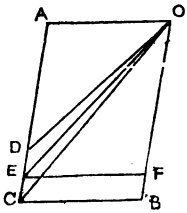
Thirdly. The direction of the resultant is that of the diagonal, in the case of any two *commensurable forces.*

This has already been shewn to be true of equal forces, or of the forces p and p; and therefore, by the proposition just proved, (making q and r both equal to p) of forces p and $2p$, therefore also of p and $3p$, and so of p and mp.

Again, the proposition being true of forces mp and p, and of forces mp and p, is true of forces mp and $2p$, therefore also of mp and $3p$, and so of mp and np, m and n being any whole numbers.

Fourthly. The proposition is also true of incommensurable forces.

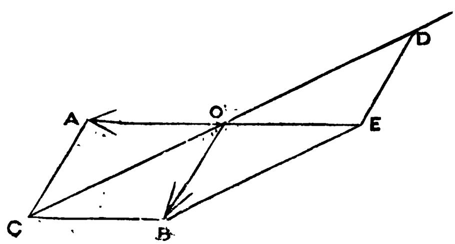
Let AO, BO represent any two incommensurable forces acting upon the point O. If their resultant do not act along the diagonal CO, suppose it to act along some other line, DO. Take any sub-multiple of the line AO that is less than DC. Mark off successive portions equal to this along the line AC, beginning at A. Some one division will ultimately fall between D and C. Let E be this point. Draw EF parallel to AO. Then AO and FO represent commensurable forces.



Combining them, we have a force acting along EO. Instead, then, of the forces AO and BO, we have a force in EO and a force in BO represented in magnitude by BF. But, by hypothesis, the resultant of the forces AO and BO is a force in DO; therefore the resultant of a force in EO and a force in BO is a force in DO, which falls without them, which is impossible. (Axiom iv.) Therefore the resultant of AO and BO cannot act along DO; and in a similar manner it may be shewn, that it cannot act along any other line above or below the diagonal CO; consequently it must act along the diagonal.

Fifthly. The diagonal represents the resultant in magnitude.

Let OA, OB represent two forces acting upon O. Complete the parallelogram AOBC. Draw the diagonal OC and produce it through O. Make OD equal to OC. Through D draw a line parallel to OB, cutting AO produced in E. Join BE, then ODEB is a parallelogram.



The resultant of OA and OB acts along OC. The forces OA and OB will consequently be kept at rest by a force in OD equal to this resultant. Suppose such a force to act in

OD, then the point O will be kept at rest; but if three forces keep a point at rest, any one must be equal and opposite to the resultant of the other two. Therefore OA must be equal and opposite to the resultant of OB and the force in OD: consequently EO must be this resultant.

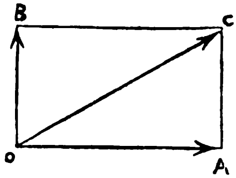
But the resultant of any two forces falls along the diagonal of the parallelogram whose sides represent these two forces in magnitude and direction. Therefore OD must represent the magnitude of the force in OD, otherwise the resultant of it and BO would not fall along OE. But the force in OD was taken equal to the resultant of the forces in AO and BO, and OD = OC = the diagonal of the parallelogram AB. Therefore the resultant of any two forces is represented in magnitude, as well as in direction, by the diagonal of the parallelogram, whose sides represent the components.

14. It follows from the preceding, that whenever the directions and magnitudes of two forces acting upon a point are known, the direction and magnitudes of their resultant can always be found by a geometrical construction. We have simply to draw from the given point two lines in the given directions, of such lengths that they shall represent the magnitudes of the two forces upon any scale whatever, and then, completing the parallelogram, the direction of the resultant is that of the diagonal drawn through the given point, and its magnitude is represented by the length of this diagonal on the same scale as the components.

It also follows that the magnitude of the resultant of two forces acting upon a point can be determined by calculation, whenever the length of the diagonal of a parallelogram can be so determined. In certain simple cases, this can be done without the aid of trigonometry. The following are some of these.

Ex. 1. To find the magnitude of the resultant when two forces, P and Q, act upon a point at angle of 90°.

Let OA and OB represent the forces P and Q. Completing the parallelograms, the diagonal OC represents the resultant.



Since the angle BOA is a right angle, the angle OAC is a right angle; therefore, (*Euclid* i. 47,)

$$OC^2 = OA^2 + AC^2.$$

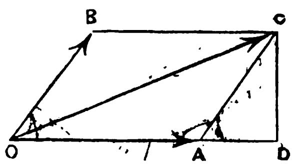
Hence, if R denote the resultant of P and Q,

$$R^2 = P^2 + Q^2,$$

or, $$R = \sqrt{(P^2 + Q^2)}.$$

Ex. 2. To find the magnitude of the resultant when two forces, P and Q, act concurrently upon a point at an angle of 60°.

Let OA and OB represent the forces P and Q. Completing the parallelogram, the diagonal OC represents the resultant or R.



Draw CD at right angles to OA produced, then, (*Euclid* ii. 12,)

$$OC^2 = OA^2 + AC^2 + 2.OA.AD.$$

But since the angle BOA is 60°, the angle CAD is 60°, and the triangle CAD is half an equilateral triangle; therefore,

$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}Q.$$

Hence, $$R^2 = P^2 + Q^2 + PQ,$$

or, $$R = \sqrt{(P^2 + Q^2 + PQ)}.$$

Ex. 3. To find the magnitude of the resultant when two forces, P and Q, act concurrently upon a point at an angle of 45°.

As in the last example, let OA and OB represent the given forces; then, as before,

$$R^2 = P^2 + Q^2 + 2P.AD.$$

But since the angles BOA is $45°$, the angle CAD is $45°$; therefore, also, ACD is $45°$. Whence $DC = AD$; consequently,

$$2AD^2 = AC^2 = Q^2;$$

$$\therefore \quad AD = \tfrac{1}{2}Q\sqrt{2}.$$

Hence, $$R^2 = P^2 + Q^2 + PQ\sqrt{2},$$

or, $$R = \sqrt{(P^2 + Q^2 + PQ\sqrt{2})}.$$

15. *If two forces,* P *and* Q, *act concurrently upon a point at any angle* θ, *and if* R *be their resultant, then* $R^2 = P^2 + Q^2 + 2PQ \cos \theta$.

For let OA and OB in the last figure represent the forces P and Q, then OC represents R.

By trigonometry, $OC^2 = OA^2 + AC^2 - 2.OA.AC \cos OAC$;

$$\therefore \quad R^2 = P^2 + Q^2 - 2PQ \cos OAC;$$

but OAC is the supplement of θ; therefore, $\cos OAC = -\cos \theta$;

$$\therefore \quad R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

The result just obtained determines the magnitude of R; to determine its direction, we have,

$$\sin COA : \sin OAC :: AC : OC;$$

therefore, if $\hat{RP}$ denote the angle between R and P,

$$\sin \hat{RP} : \sin \theta :: Q : R;$$

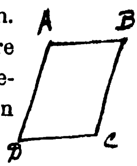
$$\therefore \quad \sin \hat{RP} = \frac{Q \sin \theta}{R}.$$

In like manner, $$\sin \hat{RQ} = \frac{P \sin \theta}{R}.$$

16. Since the components may be substituted for their resultant, their effects being equivalent, it follows from the parallelogram of forces, that for any force we may substitute two others, whose magnitudes and directions are represented by the sides of any parallelogram, which has the line representing the given force for its diagonal. And since an infinite number of parallelograms can be drawn, having a given line for their diagonal, any force can be resolved into two others in an infinite number of ways.

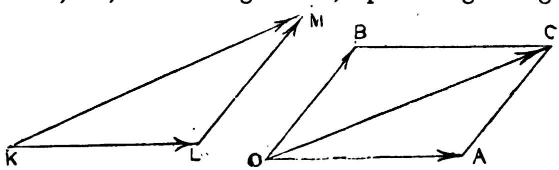
17. *If three forces* act upon a point, and their resultant be required, find the resultant of any two of them; the composition of this resultant, with the third force, will give the resultant of the three given forces. In like manner, the resultant of any number of forces acting upon a point may be found. Another, and sometimes more convenient, method is given in Article 24.

18. *Def.* The sides of any rectilinear figure are said to be taken *in order*, when taken as they would be traversed by a point moving continuously around the figure in either direction, that is to say, either as the hands of a watch revolve, or in the contrary direction. Thus, if ABCD be any quadrilateral, its sides taken in order are either AB, BC, CD, DA, or AD, DC, CB, BA. It will be sometimes convenient to describe these relatively to each other, as in direct and reverse order respectively.



19. *If two sides of a triangle, taken in order, represent in magnitude and direction two forces acting upon a point, then shall the third side, taken in reverse order, represent the resultant in magnitude and direction.*

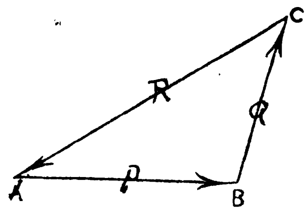
Let the sides KL, LM, of the triangle KLM, representing in magnitude and direction the forces P and Q acting at any point O,



then will KM represent in magnitude and direction the resultant of P and Q. Through O draw OA equal and parallel to KL, and pointing in the same way; also OB equal and parallel to LM, and pointing in the same way: then OA and OB represent P and Q respectively. Completing the parallelogram, the diagonal OC represents the resultant of P and Q. But since the lines OA, AC, are parallel respectively to KL, LM, and point in the same way, the angle OAC is equal to the angle KLM. Then in the triangles

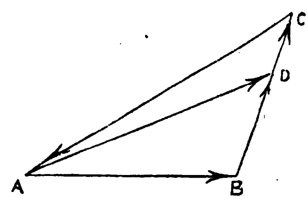
KLM, OAC, the sides KL, LM, and the angle KLM are equal severally to the sides OA, AC, and the angle OAC; therefore KM is equal to OC. It can also be easily shewn that KM is parallel to OC; therefore the line KM represents OC, the resultant of P and Q, both in magnitude and direction.

20. Triangle of Forces. *If three forces, represented in magnitude and direction by the sides of a triangle taken in order, act upon a point, they will be in equilibrium;* and, conversely, *if three forces acting upon a point, and in equilibrium, be represented in direction by the sides of a triangle taken in order, they will also be represented in magnitude by the sides of that triangle.*



First. Let three forces, represented in magnitude and direction by the sides of the triangle ABC, taken in order, that is to say, by AB, BC, and CA, act upon any point, these forces will be in equilibrium.

For, by the preceding article, the resultant of the forces represented by AB and BC is a force represented by AC; that is, a force equal and opposite to the force represented by CA. The forces are therefore in equilibrium.



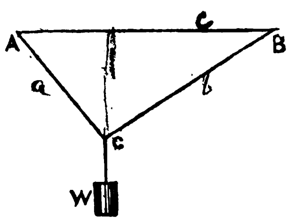
Secondly. Let any three forces, P, Q, and R, acting upon a point, be in equilibrium, and let the sides of the triangle ABC, taken in order, represent the direction of these forces, viz., AB the direction of P; BC, that of Q; and CA, that of R. Then will these sides, AB, BC, CA, represent severally the magnitudes of P, Q, and R.

Let AB represent P on any scale, then must BC represent Q on the same scale. For, if possible, let some other length BD represent Q; then, since AB, BD, two sides of a triangle taken in order, represent in magnitude and direction two forces acting upon a point, the third side AD, by Art. 19, represents their resultant. But, by hypothesis, there is equilibrium; and therefore a force represented in direction by AD is balanced by a force represented in direction by CA, which is impossible. Therefore, BD cannot represent the magnitude of Q. In like manner it may be shewn, that no other length than BC can represent Q. Then, since AB represents P, and BC represents Q, it follows that CA represents R.

The triangle whose sides are severally equally inclined to the sides of the triangle ABC, is equiangular to ABC, and consequently its sides are in the same ratio. Hence, *if three forces be in equilibrium, and any triangle be drawn whose sides are severally either parallel or equally inclined to their directions, the forces are to one another as the sides of the triangle.*

21. The following are examples of the application of the triangle of forces to the solution of mechanical problems.

Ex. 1. A weight W is sustained by two cords of given lengths CA and CB, fastened to two points A and B, lying in the same horizontal line; to determine the tensions in the cords, when the cords are at right angles.



Let a and b be the lengths of the cords CA and CB, then since ACB is a right angle, $AB = \sqrt{(a^2 + b^2)}$.

The point C is at rest under the action of three forces; viz., W acting vertically, and the tensions in CA and CB. In the triangle ACB,

AB is perpendicular to W,

BC „ the tension in CA,

CA „ the tension in CB;

$$\therefore \quad \text{tension in CA} : W :: BC : AB$$

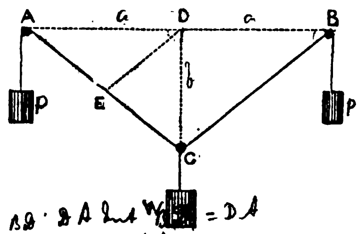
$$:: b : \surd(a^2 + b^2);$$

$$\therefore \quad \text{tension in CA} = \frac{Wb}{\surd(a^2 + b^2)}.$$

Similarly,

$$\text{tension in CB} = \frac{Wa}{\surd(a^2 + b^2)}.$$

Ex. 2. Two cords bearing equal weights, P, P, pass over pulleys placed at A and B in the same horizontal line, and are joined to a third weight W at C; what must this weight be that C may rest at a given distance below AB?



Let the vertical line through C meet the line AB in D, thence since there is no reason why C should be nearer to A than to B, AB will be bisected in D. Let AB = $2a$, and CD = b.

Through D draw DE parallel to BC, then AC is bisected in E.

In the Δ ABC, BC ∥ ED, ∴ CE : EA :: BD : DA but BD = DA ∴ CE = EA

In the triangle CDE,

DC is parallel to W,

CE „ the tension in CA,

ED „ the tension in CB.

Therefore by the triangle of forces,

$$W : \text{tension in CA} :: DC : CE.$$

But the tension in CA is equal to P, and EC = $\frac{1}{2}$AC = $\frac{1}{2}\surd(a^2 + b^2)$;

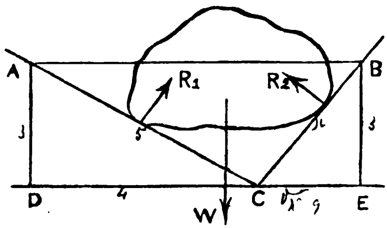
$$\therefore \quad W : P :: b : \tfrac{1}{2}\surd(a^2 + b^2);$$

$$\therefore \quad W = \frac{2Pb}{\surd(a^2 + b^2)}.$$

Ex. 3. A body whose weight is 168 lbs. rests upon two smooth inclined planes; one of the planes rises 3 in 5, and the pressure

upon it is 160 lbs.: to determine the inclination of the other plane, and also the pressure upon it.

Let AC, BC, be the two planes; the plane AC rising 3 in 5.



Take AC = 5, and through A draw AD perpendicular to CD, the horizontal line through C; then will AD = 3; and, therefore, DC $= \sqrt{(25-9)} = 4$. Through A draw B horizontally, and through B draw BE perpendicular to CE; then BE = AD = 3; and if x stand for BC, CE $= \sqrt{(x^2-9)}$.

Three forces are in equilibrium; viz., W the weight of the body, and the resistances R_1, R_2. In the triangle ABC,

AB is perpendicular to W.		
BC	,,	R_2,
CA	,,	R_1.

Therefore, by the triangle of forces,

$$R_1 : W :: CA : AB.$$

But, by hyp., $R_1 = 160$, $W = 168$, $CA = 5$, and $AB = DC + CE = 4 + \sqrt{(x^2-9)}$. Hence,

$$160 : 168 :: 5 : 4 + \sqrt{(x^2-9)};$$

$$\therefore \quad \sqrt{(x^2-9)} = \frac{5}{4};$$

$$\therefore \quad x = 3\tfrac{1}{4},$$

or the plane BC rises 3 in $3\frac{1}{4}$, or 12 in 13.

Also, $\quad R_2 : R_1 :: BC : CA;$

$\therefore \quad R_2 : 160 :: 3\frac{1}{4} : 5,$

or, $\quad R_2 = 104$ lbs.

22 *If three forces acting upon a point be in equilibrium, they are severally as the sine of the angle contained between the other two.*

Let P, Q, and R, be three forces acting upon a point, and in equilibrium. It has been shewn, in Art. 20, that if the sides of

the triangle ABC, taken in order, be severally parallel to the directions of these forces, then,

$$P : Q : R :: AB : BC : CA;$$

but, by trigonometry,

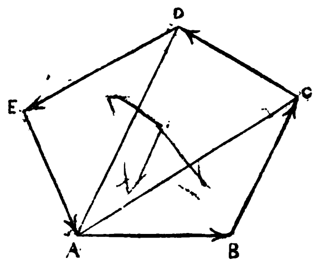
$$AB : BC : CA :: \sin BCA : \sin CAB : \sin ABC.$$

But if $\hat{PQ}$ denote the angle between P and Q, $\hat{PQ} = 180° - ABC$; and therefore $\sin ABC = \sin \hat{PQ}$. Similarly, $\sin CAB = \sin \hat{RP}$, and $\sin BCA = \sin \hat{QR}$. Therefore,

$$P : Q : R :: \sin \hat{QR} : \sin \hat{RP} : \sin \hat{PQ}.$$

23. POLYGON OF FORCES. *If any number of forces acting upon a point be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.*

Let forces represented by the sides of the polygon ABCDE, taken in order, act upon any point, they will be in equilibrium; for, by Art. 19, the resultant of the forces represented by AB and BC will be represented by AC. In like manner, the resultant of the forces represented by AC and CD will be represented by AD; or AD represents the resultant of the forces represented by AB, BC, and CD. Substituting this resultant for its components, we have remaining three forces, represented by AD, DE, EA, three sides of a triangle taken in order; and, therefore, by the triangle of forces, they are in equilibrium.

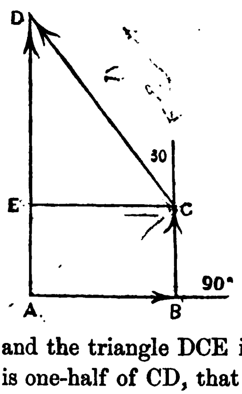


24. If any number of forces be in equilibrium, a force equal and opposite to any one will be the resultant of the remaining forces. Hence any side of a polygon, taken in reverse order, will represent the magnitude and direction of the resultant of any number of forces acting upon a point, when these forces are represented in magnitude and direction by the remaining sides of the polygon taken in direct order.

Thus, if AB, BC, CD, and DE in the last figure represent in magnitude and direction four forces acting upon any point, the remaining side AE (not EA) will represent the magnitude and direction of their resultant.

Hence, to find the magnitude and direction of any number of forces acting concurrently upon any point, draw a line parallel to one of the forces, and representing it in magnitude. Through that extremity of this line which points in the same way as the force, draw a second line parallel to the next force, pointing in the same way, and representing it in magnitude. Through the extremity of this second line, draw a third line parallel to the third force, pointing in the same way, and also representing it in magnitude. Proceed in this way until lines have been drawn representing all the forces. The straight line which completes the polygon, taken in reverse order, will represent the resultant in magnitude and direction.

25. Ex. Three forces, P, Q, and 2P, act upon a point; the angle between the first and second is 90°, and the angle between the second and third is 30°, required the magnitude of the resultant.



Through any point A, draw AB equal and parallel to P. Through B, draw BC equal and parallel to Q; and through C, draw CD equal and parallel to the third force 2P. Join A, D, then AD represents the resultant.

To find the length of AD: draw CE parallel to AB, and DE perpendicular to CE; join E, A.

By construction, ABC is 90°, and BCD is 150°; therefore the angle DCE is 60°, and the triangle DCE is half an equilateral triangle. Hence, CE is one-half of CD, that is, equals AB or P, and $DE = P\sqrt{3}$. Then, since CE is equal and parallel to AB, the figure EB is a parallelogram; and, consequently, the angle AEC is a right angle. But

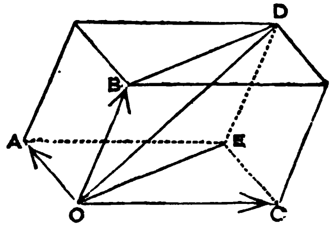
the angle DEC is also a right angle, therefore the lines AE, ED are in the same straight line. Therefore,

$$AD = AE + ED,$$
$$= Q + P\sqrt{3}.$$

26. *If three forces, acting upon a point, be represented in magnitude and direction by the three edges of a parallelopiped, their resultant will be represented in magnitude and direction by the diagonal.*

Let OA, OB, OC, the three edges of the parallelopiped BE, represent in magnitude and direction three forces acting upon O.

Since AC is a parallelogram, the diagonal OE represents the resultant of the forces OA and OC. Compound this resultant OE with the third force OB, then since BE is a parallelogram, their resultant is represented by OD; and therefore OD represents the resultant of the three forces OA, OB, OC.



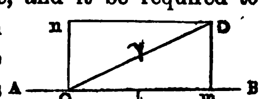
27. Lemma. *No force can produce any effect in a direction perpendicular to its own.*

For there can be no reason why it should cause the body to move towards one side rather than towards the other; and, consequently, it will not cause the body to move to either side.

28. *To determine the total effect of a force in any given direction.*

Resolve the force into two other forces, one perpendicular to, and the other acting along, the given direction. It follows from the preceding Lemma, that the last-mentioned force is the thing sought.

Thus, let CD represent a given force, and it be required to determine its total effect in the direction AB. Draw the rectangle *nm*. Then the force CD may be resolved into two forces represented by C*n* and C*m*. The force C*n*



produces no effect in the direction AB, and therefore Cm is the force required.

Let P be the given force, and let CD $= r$, and C$m = x$, then the required force : P : : x : r,

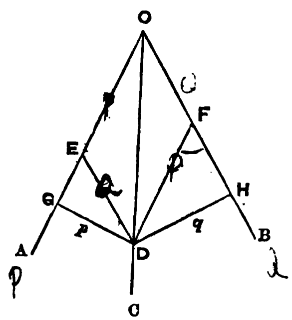
or, the required force $= \text{P}.\frac{x}{r}$.

Let the angle DC$m = \theta$, then the total effect of a force P, in a direction inclined to its own, at an angle θ, is P cos θ.

29. *Def.* The product of a force, and the perpendicular distance of its direction from any given point, is termed the *moment* of the force about that point.

30. *The moments of two converging forces about any point in the direction of their resultant are equal;* and, conversely, *if the moments of two converging forces about any point in their plane be equal, that point will lie in the direction of the resultant*

Let P and Q be two forces acting respectively in the lines AO and BO, and let CO be the direction of their resultant. In CO take any point D, and from D draw the perpendiculars DG and DH. Let DG $= p$ and DH $= q$, then shall Pp = Qq.



Complete the parallelogram EF. The triangles DEG, DFH will be similar; and, therefore,

$$p : q :: \text{DE} : \text{DF},$$

or, $$:: \text{DE} : \text{OE}.$$

But by the triangle of forces, Q : P : : DE : OE; therefore,

$$\text{Q} : \text{P} :: p : q,$$

and $\therefore$ $$\text{P}p = \text{Q}q$$

Again, let the moments of P and Q about the point D be equal, that point will lie in the direction of their resultant. Let the same construction remain. Then because $Pp = Qq$,

$$Q : P :: p : q.$$

But, because the triangles DEG, DFH are similar, $p : q ::$ DE : DF,

$$\therefore \qquad Q : P :: DE : DF,$$

$$\text{or,} \qquad Q : P :: OF : OE.$$

Then, since OF, OE, two sides of a parallelogram, represent P and Q in magnitude and direction, OD represents their resultant, and therefore D is a point in the direction of the resultant.

31. The propositions established in the last section enable us to decide whether the resultant of two converging forces does or does not pass through any given point, and hence furnish us with a ready means for the determination of the equilibrium of a body, moveable only about a fixed point, when acted on by two converging forces. In the case of such a body, it is clear that no motion can result from the action of any force whose line of direction passes through the fixed point. The effect of such a force is simply to press the body against the fixed point. It hence follows that no motion will ensue when two or more forces act upon the body, if their resultant passes through the fixed point. This we have seen will happen in the case of two converging forces, when their moments about that point are equal, and only then. This, therefore, is the necessary and sufficient condition of the equilibrium of a body, moveable about a fixed point, when acted on by two converging forces.

In the following chapter, a similar principle will be established for any number of forces, whether converging or parallel.

EXAMPLES.

i. *Problems not requiring a knowledge of Trigonometry.*

1. Two forces of 7 and 33 lbs. act upon a point at an angle of 60°, required the resultant.
The resultant is 37 lbs.

2. Two forces of 5 and 21 lbs. act upon a point at an angle of 120°, required their resultant.
The resultant is 19 lbs.

3. The resultant of two equal forces acting upon a point at an angle of 90° is 100 lbs., required the value of each component.
Each component is 70·7107 lbs.

4. The resultant of two equal forces acting upon a point at an angle of 135° is 10 lbs, required the value of each component.
Each component is 13·0659 lbs.

5. The resultant of two forces acting upon a point at an angle of 150° is 19 lbs., one of the components is 11 lbs., find the other component.
The component required is $16\sqrt{3}$ lbs.

6. Two forces of 9 lbs. and 56 lbs. acting upon a point are balanced by a force of 61 lbs., show that the angle between the two forces is 60°.

7. Two forces of 7 lbs. and 40 lbs. acting upon a point are balanced by a force of 37 lbs., show that the angle between the two forces is 120°.

8. Two forces acting upon a point at an angle of 120° have a

resultant whose magnitude is $P\sqrt{3}$, and which makes an angle of $30°$ with one of the forces, find the magnitude of the two forces.

The two forces are P and 2P.

9. Three equal forces acting upon a point are in equilibrium, show that the angle between any two of them is $120°$.

10. If two forces P and 3P act upon a point at an angle of $120°$, show that the magnitude of the resultant is not affected by doubling the smaller force.

11. Five equal forces act upon a point in such a way that the first is at right angles with the second; the third is at right angles with the resultant of the first and second; the fourth, at right angles to the resultant of the first, second, and third; and the fifth, at right angles with the resultant of the other four; required the magnitude of the resultant of the five forces.

If P be the magnitude of each of the components, the resultant will be equal to

$$P\sqrt{5}.$$

X 12. If six equal forces act upon a rigid body along the sides of a regular hexagon, in such a way that the forces act concurrently at each angle, show that the forces will be in equilibrium.

13. Three forces, whose magnitudes are severally 16, 63, and 156 lbs., act upon a point at right angles to each other, required the magnitude of their resultant.

Show by the parallelopiped of forces that the resultant is 169 lbs.

14. Three forces, of 9, 15, and 28 lbs. act upon a point concurrently; the angle between the first and second is $60°$, and the third is at right angles with each of the other two: required the magnitude of the resultant.

The resultant is 35 lbs.

15. A smooth ring, sustaining a weight W, is carried by a cord fastened at two points, A and B, in the same horizontal line; required the tension in the cord.

Let $2b$ be the length of the cord, and $2a$ the distance AB, then the tension in the cord will be equal to

$$\frac{Wb}{2\sqrt{(b^2 - a^2)}}$$

16. Required the horizontal force necessary to draw, on a level road, a carriage wheel over an obstacle of given height.

Let W be the weight of the carriage, a the radius of the wheel, and c the height of the obstacle, then the required force will be equal to

$$\frac{W.\sqrt{(2ac - c^2)}}{a - c}.$$

17. Two strings, fastened to two pegs, A and B, in a vertical wall, of which A is the higher, are joined at the point C to a weight W; the strings are of such lengths that BC is horizontal, and the angle BCA is 135°: find the tensions in the strings.

The tension in AC is equal to $W\sqrt{2}$, and the tension in BC is equal to W.

18. Four equal forces act concurrently upon a point, and in the same plane, the angle between the first and second, that between the second and third, and that between the third and fourth, are each equal to 72°: find the magnitude of the resultant.

The resultant is equal to one of the components.

+ 19. A ball, whose weight is W, slides along a smooth rod, inclined at an angle of 30° with the vertical line; what force applied in the direction of the rod will keep the ball at rest, and what is the pressure upon the rod?

The required force is equal to $\frac{1}{2}W\sqrt{3}$, and the pressure upon the rod is equal to $\frac{1}{2}W$.

20. A heavy ball, which slides along a smooth rod, inclined at an angle of 45° with the vertical line, is kept at rest by a force P acting along the rod; required the weight of the ball, and the pressure upon the rod.

The weight of the ball is equal to $P\sqrt{2}$, and the pressure upon the rod is equal to P.

ii. *Problems requiring a knowledge of Trigonometry.*

21. Two forces of 65 and 84 lbs. act upon a point at an angle of 70°, required the magnitude of the resultant.

The resultant is 122·54 lbs.

22. Two forces of 20 and 64 lbs. act upon a point; their resultant is 76 lbs., what is the angle between the two forces?

The angle between the forces is 60°.

23. Two forces of 3 and 5 lbs. act upon a point; their resultant is 7 lbs., what is the angle between the resultant and the larger of the two forces?

The angle required is equal to

$$\sin^{-1}\left(\frac{3\sqrt{3}}{14}\right)$$

24. One of two components is double the other, and their resultant is equal to half their sum; required the angle between them.

The angle required is equal to

$$\cos^{-1}\left(-\frac{11}{16}\right)$$

25. The resultant of two forces acting upon a point is equal to one half of one of them, and is at right angles with the other; required the angle between the two forces.

The angle required is 150°.

26. If X, Y, and Z be three forces acting upon a point, and the angle between X and Y be equal to α, and the angle between Y and Z be equal to β, and if R be the resultant, then

$$R^2 = X^2 + Y^2 + Z^2 + 2XY \cos \alpha + 2YZ \cos \beta + 2ZX \cos (\alpha + \beta).$$

27. A weight W, to which two cords are attached, rests upon a smooth horizontal table; one cord passes over a pulley placed at the edge of the table, and bears a weight Q; the other cord passes over a pulley placed above the table, and bears a weight P; required the inclination of the latter cord to the table?

The inclination required is equal to

$$\cos^{-1}\left(\frac{Q}{P}\right).$$

28. One end of a cord is fastened to a fixed point A, the other is carried over a fixed pulley at B, and fastened to a weight P; what weight must be attached to the cord at a point C, midway between AB, that the part AC may rest horizontally?

Let AB be inclined to the horizon at an angle α, then the required weight will be equal to

$$\frac{2P \sin \alpha}{\sqrt{(5 - 4 \cos \alpha)}}.$$

29. In the preceding, show that the tension in AC is equal to

$$\frac{P(1 - 2 \cos \alpha)}{\sqrt{(5 - 4 \cos \alpha)}}.$$

30. A particle, whose weight is W, rests against the circumference of a circular plate, whose plane is vertical; a cord attached to the particle passes over a pulley placed vertically above the highest point of the circle, at a distance from the circle equal to the radius; required the position in which the particle will rest, and the pressure on the circle, when a weight P hangs from the cord.

In the position of equilibrium, the inclination of the cord to the vertical line is equal to

$$\cos^{-1}\left(\frac{3W^2 + 4P^2}{8WP}\right),$$

and the pressure on the circle is equal to

$$\tfrac{1}{2}W.$$

31. The sum of the moments of two converging forces about *any* point in their plane is equal to the moment of their resultant.

Let P and Q be forces acting on the point O, and let R be the resultant, making the angles α and β with P and Q respectively; then, by the parallelogram of forces, the relations between P, Q, and R may be expressed by the equations

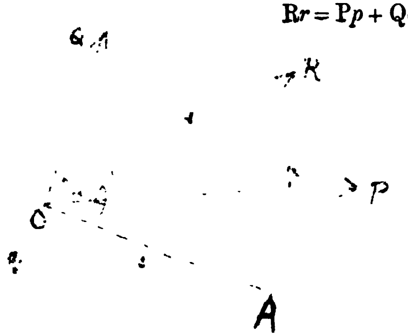
$$R = P\cos\alpha + Q\cos\beta, \qquad \text{(i.)}$$
$$0 = P\sin\alpha - Q\sin\beta. \qquad \text{(ii.)}$$

Let A be any point in the plane of the forces, and let the line AO be equal to c, and be inclined to the direction of R at an angle θ. Multiply i. by $c\sin\theta$, and ii. by $c\cos\theta$, and subtract; then

$$Rc\sin\theta = Pc\sin(\theta-\alpha) + Qc\sin(\theta+\beta);$$

whence, if p, q, and r denote the perpendiculars from A to P, Q, and R respectively,

$$Rr = Pp + Qq.$$

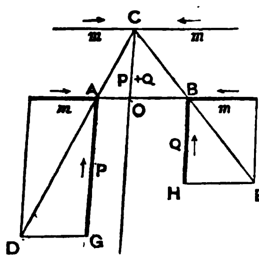


CHAPTER III.

ON PARALLEL FORCES.

32. *To determine the magnitude and position of the resultant of two concurrent parallel forces* P *and* Q.

In the directions of P and Q take <u>any</u> points A and B, and introduce at A and B two equal and opposite forces (m, m). The resultant of m and P is AD, and the resultant of m and Q is BE. Remove the points of application of these two resultants from A and B to C, and resolve them into their original components. We shall then have two equal and opposite forces m, m, which will destroy each other, and two concurrent forces P and Q acting along CO. The resultant therefore is a force whose magnitude is P + Q acting along CO.



Because the triangles DGA, AOC are similar, $m : P :: AO : OC$;

$$\therefore \qquad P \times AO = m \times OC.$$

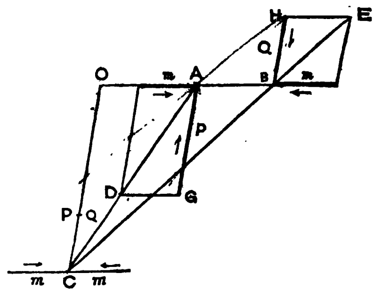
$$\text{Similarly,} \qquad Q \times BO = m \times OC,$$

$$\therefore \qquad P \times AO = Q \times BO.$$

33. *To find the resultant of two non-concurrent parallel forces* P *and* Q.

As in the former case, take <u>any</u> two points A and B in the directions of P and Q, and introduce two equal and opposite forces (m, m).

Compound these with P and Q. Produce the directions of the resultants till they meet in C. Remove the point of application of these resultants from A and B to C, and resolve them into their original components. We shall then have two equal and opposite forces (m, m) acting in a line parallel to AB, which will destroy each other, and along CO parallel to the directions of P and Q, two non-concurrent forces equal severally to P and Q. The resultant then will be a force $P - Q$ acting along CO.



Because the triangles DGA, AOC are similar, $m : P :: AO : OC$;

$$\therefore \qquad P \times AO = m \times OC.$$

$$\text{Similarly,} \qquad Q \times BO = m \times OC,$$

$$\therefore \qquad P \times AO = Q \times BO.$$

34. From these investigations we see, in the case of two parallel forces;—That the resultant is always parallel to the components;—That when the components are concurrent, the resultant is concurrent with them, and equal to their sum; but when the components are non-concurrent, the resultant is equal to their difference, and concurrent with the greater;—That if any line whatever (for the points A and B were taken quite arbitrarily) be drawn across the directions of the components, it will be cut by the resultant at a point such that the one force multiplied into its distance from the resultant measured along this line, equals the other force multiplied into its distance.

35. In Art. 33, if the forces P and Q had been equal, the construction would have been impossible. The lines AD, BE would have been parallel, and consequently the determination of the

point C would have become an impossibility. The two forces will therefore have no resultant. Such a combination of forces is termed a *couple.*

36. *The resultant of any number of concurrent parallel forces equals the sum of the components.*

Let P_1, P_2, P_3, P_4, be the forces. Let R_1 be the resultant of P_1 and P_2, then

$$R_1 = P_1 + P_2.$$

Let R_2 be the resultant of R_1 and P_3; then

$$\begin{aligned} R_2 &= R_1 + P_3, \\ &= P_1 + P_2 + P_3. \end{aligned}$$

The resultant of R_2 and P_4 will be the resultant of the given forces; let this be R, then

$$\begin{aligned} R &= R_2 + P_4, \\ &= P_1 + P_2 + P_3 + P_4; \end{aligned}$$

and similarly of any other number of parallel forces.

37. Hence the resultant of any number of parallel forces equals their algebraic sum.

Let P_1, P_2, &c. and Q_1, Q_2, &c. be any number of parallel forces, of which P_1, P_2, &c. act in one direction, and Q_1, Q_2, &c. in the opposite direction. Let X be the resultant of P_1, P_2, &c. and Y the resultant of Q_1, Q_2, &c. and let X be greater than Y. Then, since P_1, P_2, &c. are concurrent forces,

$$X = P_1 + P_2 + \&c$$

For a similar reason,

$$Y = Q_1 + Q_2 + \&c.$$

If R be the resultant of the given forces, R must be the resultant of X and Y; but these are non-concurrent forces, therefore

$$R = X - Y,$$

$$\therefore \quad R = P_1 + P_2 + \&c. - Q_1 - Q_2 - \&c.$$

38. *Two concurrent parallel forces,* P *and* Q, *act at a distance* **a** *from each other, to find the distance of the resultant from either force.*

Let x be the distance of the resultant from P; then, since the resultant passes between P and Q, its distance from Q will be $a-x$. Therefore

$$Px = Q\ (a-x),$$

or, $$(P+Q)\ x = Qa,$$

$$\therefore \qquad x = \frac{Qa}{P+Q} = \frac{Qa}{R},$$

or the distance of the resultant from one of the components equals the distance between the components multiplied by the other component, and divided by the resultant.

39. *Two non-concurrent forces,* P *and* Q, *act at a distance* **a** *from each other, to find the distance of the resultant from either force.*

Let P be the greater force, and R their resultant, then R equals $P-Q$, and passes outside P. Let x be the distance of P from the resultant, then $a+x$ is the distance of Q. Therefore

$$Px = Q\ (a+x),$$

or, $$(P-Q)\ x = Qa,$$

$$\therefore \qquad x = \frac{Qa}{P-Q} = \frac{Qa}{R},$$

or, as before, the distance of the resultant from either component equals the distance between the components multiplied by the other component, and divided by the resultant.

40. *To resolve a given force* P *into two parallel forces, acting at given distances on each side of it.*

Let a and b be the given distances, and let X and Y be the forces required, then P will be the resultant of X and Y; consequently $X+Y=P$, and $aX=bY$;

$$\therefore \qquad bX + bY = Pb,$$

or, $$bX + aX = Pb,$$

$$\therefore \qquad X = P\frac{b}{a+b}, \text{ and } \therefore\ Y = P\frac{a}{a+b}.$$

Hence, to find either component, multiply the given force by the distance of the other component, and divide by the distance between the two components.

41. *To resolve a given force* P *into two parallel forces, acting at given distances on the same side of it.*

Let a and b be the given distances, a being the greater, and X and Y the required forces. From Art. 34, it appears that X and Y will be non-concurrent, and that the force nearer to P, namely, Y, will act concurrently with P. Also, $Y - X$ equals P, and aX equals bY;

$$\therefore \qquad bY - bX = Pb,$$

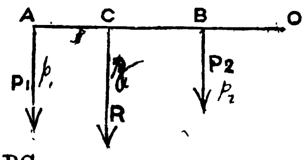
$$\text{or,} \qquad aX - bX = Pb,$$

$$\therefore \qquad X = P\frac{b}{a-b}, \text{ and } \therefore Y = P\frac{a}{a-b};$$

or, as before, to find either component, multiply the given force by the distance of the other component, and divide by the distance between the two components.

42. *The sum of the moments of any two concurrent parallel forces about any point in their plane, lying without them, is equal to the moment of the resultant about that point.*

Let P_1 and P_2 be any two concurrent parallel forces, and let R be their resultant; let p_1, p_2, and r be their distances respectively from the point O; then shall



$$P_1p_1 + P_2p_2 = Rr.$$

For, since R is the resultant of P_1 and P_2,

$$P_1 \times AC = P_2 \times BC;$$

$$\therefore \qquad P_1(p_1 - r) = P_2(r - p_2),$$

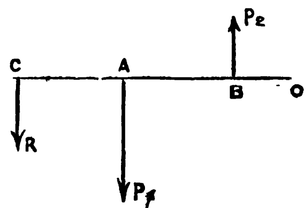
$$\text{or,} \qquad P_1p_1 + P_2p_2 = (P_1 + P_2)r.$$

But $R = P_1 + P_2$; therefore,

$$P_1p_1 + P_2p_2 = Rr.$$

43. *The difference of the moments of any two non-concurrent parallel forces about any point in their plane, lying without them, is equal to the moment of the resultant about that point.*

Let P_1 and P_2 be any two non-concurrent parallel forces, R their resultant, and p_1, p_2, r their distances respectively from the point O; then shall



$$P_1p_1 - P_2p_2 = Rr.$$

For, since R is the resultant of P_1 and P_2,

$$P_1 \times AC = P_2 \times BC;$$

$$\therefore \quad P_1(r-p_1) = P_2(r-p_2);$$

$$\text{or,} \quad (P_1 - P_2)\, r = P_1p_1 - P_2p_2.$$

But $R = P_1 - P_2$; therefore,

$$Rr = P_1p_1 - P_2p_2.$$

44. *If any number of concurrent parallel forces act in the same plane, the sum of their moments about any point in the plane lying without them is equal to the moment of the resultant about that point.*

Let the forces be P_1, P_2, P_3, P_4, &c., and their perpendicular distances from the given point p_1, p_2, p_3, p_4, &c., respectively. Let R_1 be the resultant of P_1 and P_2, and its distance from the given point r_1, then, by Article 42,

$$R_1r_1 = P_1p_1 + P_2p_2.$$

Let R_2 be the resultant of R_1 and P_3, and its distance from the given point r_2;

$$\therefore \quad R_2r_2 = R_1r_1 + P_3p_3,$$
$$= P_1p_1 + P_2p_2 + P_3p_3,$$

and similarly for the rest; and hence if R be the resultant of all the forces, and its distance from the given point be r, then

$$Rr = P_1p_1 + P_2p_2 + P_3p_3 + P_4p_4 + \&c.$$

45. *If any number of parallel forces act in the same plane, the algebraic sum of their moments about any point in the plane lying without them is equal to the moment of the resultant about that point.*

Let P_1, P_2, &c., Q_1, Q_2, &c., be any number of parallel forces acting in the same plane, of which P_1, P_2, &c. act in one direction, and Q_1, Q_2, &c. act in the contrary direction. Let p_1, p_2, &c. and q_1, q_2, &c. denote the perpendicular distances of the forces from the given point. Let X be the resultant of P_1, P_2, &c. and Y that of Q_1, Q_2, &c., and let x and y be the perpendicular distances of X and Y from the given point; then, by the preceding article,

$$Xx = P_1p_1 + P_2p_2 + \&c.,$$

and

$$Yy = Q_1q_1 + Q_2q_2 + \&c.$$

But if R be the resultant of all the forces, it is also the resultant of X and Y; and hence, if r be the distance of R from the given point, by Art. 43,

$$Rr = Xx - Yy;$$

and, therefore,

$$Rr = P_1p_1 + P_2p_2 + \&c. - Q_1q_1 - Q_2q_2 - \&c.$$

46. *If any number of parallel forces act in the same plane, the algebraic sum of their moments about any point in their resultant will be equal to zero.*

Let P_1, P_2, &c. be forces lying on one side of the resultant, and p_1, p_2, &c. their perpendicular distances from any given point in the resultant, and let Q_1, Q_2, &c. denote the forces on the other side, and q_1, q_2, &c. their distances from the same point. Then shall

$$P_1p_1 + P_2p_2 + \&c. - Q_1q_1 - Q_2q_2 - \&c. = 0.$$

For, let P be the resultant of the forces P_1, P_2, &c., and let p be its distance from the given point, then since this point lies without these forces, and in the same plane, it follows, from Art. 45, that

$$Pp = P_1p_1 + P_2p_2 + P_3p_3 + \&c.$$

Similarly, if Q be the resultant of Q_1, Q_2, &c., and q its distance from the given point,

$$Qq = Q_1q_1 + Q_2q_2 + Q_3q_3 + \&c.$$

But the resultant of the given forces is the resultant of P and Q, and therefore $Pp = Qq$.

$$\therefore \quad P_1p_1 + P_2p_2 + \&c. = Q_1q_1 + Q_2q_2 + \&c.$$

or,

$$P_1p_1 + P_2p_2 + \&c. - Q_1q_1 - Q_2q_2 - \&c. = 0.$$

A force which acts on one side of the given point tends to cause revolution about that point in a direction the opposite to that

which it tends to produce when acting on the other side; the moments $Q_1 q_1$, &c. must therefore be taken with the contrary sign to that of $P_1 p_1$, &c.; and hence the equation just deduced shows that the algebraic sum of the moments is equal to zero.

47. Conversely, *if the algebraic sum of the moments of any number of parallel forces acting in the same plane, and not in equilibrium, about any point in the plane, be equal to zero, that point is in the direction of the resultant.*

Let P_1, P_2, &c. be the forces, and p_1, p_2, &c. their distances from a point in their resultant.

If possible, let the sum of the moments about a point not in the resultant be equal to zero; then, if x be the distance of this point from the resultant, $p_1 - x$, $p_2 - x$, &c. will be the distances of the forces from this point; and therefore, by hypothesis,

$$P_1(p_1 - x) + P_2(p_2 - x) + \&c. = 0.$$

But, by the preceding article,

$$P_1 p_1 + P_2 p_2 + \&c. = 0;$$

therefore, subtracting,

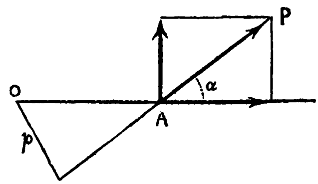
$$(P_1 + P_2 + \&c.)x = 0;$$

or,

$$Rx = 0.$$

But, by hypothesis, the forces are not in equilibrium, that is, R does not vanish; and therefore $x = 0$, or the assumed point must be in the resultant.

48. *The resultant of any number of converging forces acting in the same plane passes through a given point in their plane, when the sum of the moments about that point is equal to zero.*



Let P_1, P_2, &c. be any number of converging forces acting in the same plane, and p_1, p_2, &c. be the perpendicular distances of their lines of direction from any point O. Let a_1, a_2, &c. be the angles which the directions of the forces make with any line passing through O. Let the direction of

P_1 meet this line at A_1. Resolve the force P_1 into two forces, one acting along, and the other at right angles to A_1O; these will be $P_1 \cos a_1$, and $P_1 \sin a_1$ respectively. In like manner resolve the other forces. The several components acting along A_1O have no tendency to produce motion about O, and may therefore be disregarded. The other components, being all at right angles with OA, are parallel forces, and therefore, by the preceding, their resultant passes through O, if

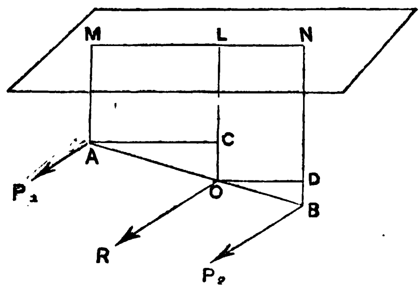
$$P_1 \sin a_1 . A_1O + P_1 \sin a_2 . A_2O + \&c. = 0.$$

But $A_1O \sin a_1 = p_1$; and, similarly, $A_2O \sin a_2 = p_2$, and so on. Therefore the resultant passes through O, if

$$P_1 p_1 + P_2 p_2 + \&c. = 0.$$

49. *Def.* The product of a force, and the perpendicular distance of its point of application from any given plane, is termed the moment of the force with respect to the plane.

50. *If any two concurrent parallel forces have their points of application on the same side of a given plane, the sum of the moments of the forces with respect to the plane is equal to the moment of the resultant.*



Let A and B be the points of application of any two concurrent parallel forces, P_1 and P_2, and let R their resultant meet AB in O.

Let perpendiculars drawn from A, B, and O meet the given plane in M, N, and L. Let AM $= h_1$, BN $= h_2$, and OL $= h$. Then shall $Rh = P_1h_1 + P_2h_2$. Draw AC and OD parallel to MN.

Then, since R is the resultant of the two concurrent parallel forces, P_1 and P_2,

$$R = P_1 + P_2;$$
$$\therefore \quad Rh = P_1h + P_2h,$$
$$= P_1(h_1 + OC) + P_2(h_2 - BD),$$
$$= P_1h_1 + P_2h_2 + P_1 . OC - P_2 . BD.$$

But, since the resultant of P_1 and P_2 passes through O,

$$P_1 : P_2 :: BO : AO;$$

and, since the triangles ACO, ODB are similar,

$$BO : AO :: BD : OC;$$

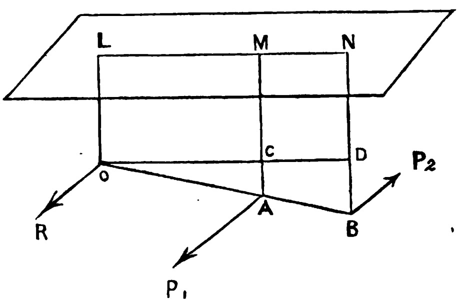
therefore, $\quad P_1 : P_2 :: BD : OC,$

or, $\quad P_1 . OC = P_2 . BD.$

Hence, $\quad Rh = P_1h_1 + P_2h_2.$

51. *If any two non-concurrent parallel forces have their points of application on the same side of a given plane, the difference of the moment of the forces with respect to the plane is equal to the moment of the resultant.*

Let A and B be the points of application of two non-concurrent parallel forces, P_1 and P_2, of which P_1 is the greater; and let R, their resultant, meet AB produced in O. Let perpendiculars from A, B, and O meet the given plane in M, N, and L. Let $AM = h_1$, $BN = h_2$, and $OL = h$; then shall $\quad Rh = P_1h_1 - P_2h_2$.



Draw OD parallel to LN. Then, since R is the resultant of two non-concurrent parallel forces,

$$R = P_1 - P_2;$$
$$\therefore \quad Rh = P_1h - P_2h,$$
$$= P_1(h_1 - AC) - P_2(h_2 - BD),$$
$$= P_1h_1 - P_2h_2 - P_1 . AC + P_2 . BD.$$

But, since the resultant of P_1 and P_2 passes through O,

$$P_1 : P_2 :: BO : AO;$$

and, since the triangles AOC, BOD are similar,

$$BO : AO :: BD : AC;$$

therefore, $\quad P_1 : P_2 :: BD : AC;$

or $\quad P_1 . AC = P_2 . BD;$

and hence, $\quad Rh = P_1h_1 - P_2h_2.$

52. *If any number of concurrent parallel forces have their points of application all on the same side of a given plane, the sum of the moments of the forces with respect to the plane is equal to the moment of the resultant.*

Let the forces be P_1, P_2, P_3, &c. acting at points whose perpendicular distances from the plane are h_1, h_2, h_3, &c.

Let the resultant of P_1 and P_2 be R_1; let it act at a distance z_1 from the given plane; then, by Article 50,

$$R_1z_1 = P_1h_1 + P_2h_2.$$

Let the resultant of R_1 and P_3 be R_2; and let it act at a distance z_2 from the given plane; then,

$$\begin{aligned} R_2z_2 &= R_1z_1 + P_3h_3, \\ &= P_1h_1 + P_2h_2 + P_3h_3; \end{aligned}$$

and in like manner for the rest. Hence, if R be the resultant of all the forces, and h the distance of its point of application from the given plane,

$$Rh = P_1h_1 + P_2h_2 + P_3h_3 + P_4h_4 + \&c.$$

53. *The algebraic sum of the moment of any number of parallel forces, with respect to a plane lying without their points of application, is equal to the moment of the resultant.*

For, let the notation P_1, P_2, h_1, h_2, &c. apply to forces acting in one direction; and Q_1, Q_2, k_1, k_2, &c. to those acting in the opposite direction. Let R_1, R_2, acting at distances z_1, z_2, be the respective resultants of each set of forces; then,

$$R_1z_1 = P_1h_1 + P_2h_2 + \&c.$$

$$R_2z_2 = Q_1k_1 + Q_2k_2 + \&c.$$

Let R be the general resultant, acting at a distance h; then, since R_1 and R_2 are non-concurrent (Art. 51), $Rh = R_1z_1 - R_2z_2$;

$$\therefore \quad Rh = P_1h_1 + P_2h_2 + \&c. - Q_1k_1 - Q_2k_2 - \&c.$$

54. *The algebraic sum of the moments of any number of parallel forces, with respect to any plane whatever, is equal to the moment of the resultant with respect to that plane.*

Let the forces P_1, P_2, &c. act on one side of the plane, at points distant severally from the plane h_1, h_2, &c. Let Q_1, Q_2, &c. k_1, k_2, &c. apply to the forces on the other side of the plane. Let R, their resultant, fall on the same side of the given plane as the forces P_1, P_2, &c. at a distance h.

Let a second plane be drawn parallel to the given plane, beyond the points of application of the forces Q_1, Q_2, &c. and let x be the distance between the two planes. Then the distances of the points of application of the given forces from the second plane will be $h_1 + x$, $h_2 + x$, &c. $x - k_1$, $x - k_2$, &c. and of the resultant $h + x$. Hence, by the preceding Article,

$$R(h + x) = P_1(h_1 + x) + P_2(h_2 + x) + \&c.$$
$$+ Q_1(x - k_1) + Q_2(x - k_2) + \&c.$$

But $R = P_1 + P_2 + \&c. + Q_1 + Q_2 + \&c.$ and therefore

$$Rx = P_1x + P_2x + \&c. + Q_1x + Q_2x + \&c.;$$

$$\therefore \quad Rh = P_1h_1 + P_2h_2 + \&c. - Q_1k_1 - Q_2k_2 - \&c.$$

55. In the preceding, if the plane pass through the point of application of the resultant, $h = 0$; hence, in such a case, the algebraic sum of the moment of the forces equals zero.

EXAMPLES.

i. *Problems not requiring a knowledge of Trigonometry.*

1. A bar AB, 5 feet 4 inches in length, rests horizontally upon two vertical props A and B; a weight of 320 lbs. is suspended from the bar at a distance of 27 inches from A: required the pressures upon the props.

The pressure on A is 185 lbs., and the pressure on B is 135 lbs.

2. Two weights P and Q, of which P is the greater, are suspended from the extremities of a straight bar; where must a third weight W be attached that the bar may balance about its middle point?

Let $2a$ be the length of the bar, then the weight W must be attached at a distance from the middle point, on the side of Q, equal to

$$\frac{(P-Q)\,a}{W}.$$

3. Three concurrent parallel forces, of 15, 20, and 24 lbs., act in the same plane upon a rigid body, and at distances of 12, 14, and 30 inches respectively to the right of a certain point in their plane: required the distance of their resultant from the same point.

The required distance is 20 inches.

4. If a weight W rests upon a triangular table ABC, at a point O, show that the pressure on the leg A is to W as the triangle BOC is to the triangle ABC.

5. If the magnitude of four concurrent parallel forces be 8, 12, 15, and 25 respectively, and the distances of their points of

application from a given plane be 5, 10, 8, and 6, what is the distance of the point of application of their resultant from the same plane?

The required distance is $7\frac{1}{8}$.

6. If the distances of the points of application of n equal concurrent parallel forces from a plane passing through the point of application of the resultant form a descending arithmetical series, of which the first term is a, and common difference b, show that

$$\frac{a}{b}=\frac{n-1}{2}.$$

7. Resolve a given force R into three parallel components, P_1, P_2, P_3, acting at given points. Let z be the distance of the point of application of R from any line in the plane of the points, and z_1, z_2, z_3, the distances of the given points from the same line, and let h, h_1, h_2, h_3, be the respective distances of the same points from any other line; then shall

$$P_1=R\frac{(z\,h_2-h\,z_2)+(z_2h_3-h_2z_3)+(z_3h\;-h_3z\,)}{(z_1h_2-h_1z_2)+(z_2h_3-h_2z_3)+(z_3h_1-h_3z_1)}.$$

8. Show that the resolution of any force into more than three parallel components, acting at given points in one plane, is an indeterminate problem.

9. A bar is formed of unequal lengths of two substances, each of uniform thickness and density, the one weighing m oz. per inch, and the other n oz. per inch; how far from the middle point must a fulcrum be placed that the bar may balance about it by its own weight?

Let a be the length in inches of the one part, and b that of the other, then the distance required is

$$\frac{ab}{2}\cdot\frac{m\sim n}{ma+nb}.$$

10. A series of n equal weights is suspended from a weightless bar, in such a way that the distance between the first and second is a, that between the second and third $2a$, that between the third and fourth $3a$, and so on, the successive intervals forming an arithmetic series; required the distance of the resultant from the first weight.

Assuming that the sum of n terms of the series $1 + 3 + 6 + 10 + \&c. + \frac{n.(n+1)}{2}$ is $\frac{n.(n+1)(n+2)}{6}$, show that the distance required is equal to

$$\frac{(n^2-1)a}{6}.$$

11. Weights P, 2P, 3P, &c....nP are suspended at distances a, $2a$, $3a$, &c....na from one extremity of a weightless rod; required the distance of the resultant from that extremity.

The distance required is

$$\frac{(2n+1)a}{3}.$$

12. A straight weightless bar ABCD rests horizontally upon two pegs at B and C; required the pressures upon the pegs, when weights P and Q are suspended from A and D.

Let $AB = a$, $BC = c$, and $CD = b$; then the pressure at B is equal to

$$\frac{P(a+c) - Qb}{c},$$

and the pressure at C is equal to

$$\frac{Q(b+c) - Pa}{c}.$$

ii. *Problems requiring a knowledge of Trigonometry.*

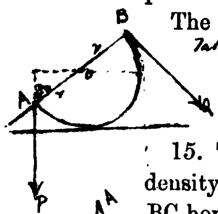
13. A weightless bar ACB, bent so as to form a right angle at C, is moveable about a fixed axis at C; required the position of equilibrium, when weights P and Q are suspended from A and B.

Let AC $= a$, and BC $= b$; then, if θ be the inclination of AC to the horizontal line,

$$\tan\theta = \frac{Pa}{Qb}.$$

14. Weights P and Q, of which P is the greater, are suspended by cords from A and B, opposite points in the rim of a weightless hemispherical bowl, resting upon a horizontal table; required the position of equilibrium, both weights hanging outside of the bowl.

The inclination of AB to the vertical line is equal to



Take moments about ∅.

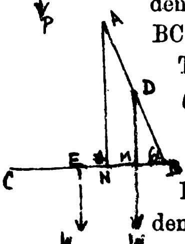
$P r \sin\theta = Q r$

$\therefore \theta = \sin^{-1}\left(\frac{Q}{P}\right)$

$$\text{Sin}^{-1}\left(\frac{Q}{P}\right).$$

15. Two equal bars, AB and BC, of uniform thickness and density, rigidly connected at B, when suspended from A, rest with BC horizontal; required the angle ABC.

The angle required is equal to



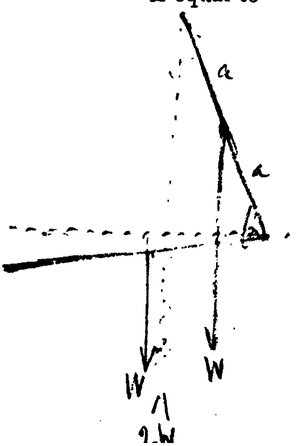
$\cos\alpha = \frac{MB}{BD} = \frac{MB}{BE}$

$$\cos^{-1}\left(\frac{1}{3}\right).$$

16. Two equal bars, AB and BC, of uniform thickness and density, rigidly connected at B, are suspended from A; required the inclination of the bar AB to the vertical line through A, in the position of equilibrium.

Let the angle ABC be equal to α; then the inclination required is equal to

$$\cot^{-1}\left(\frac{3 - \cos\alpha}{\sin\alpha}\right).$$



$2W \cdot 2a \sin\theta = W a \sin\theta + W a \cos(\alpha - 90 - \theta)$

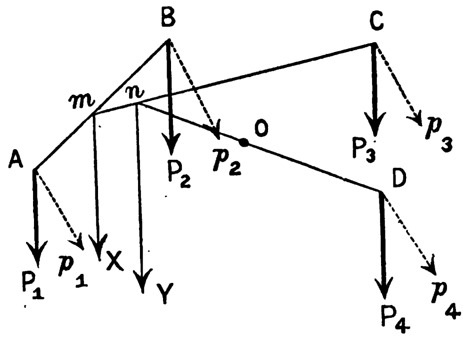
$\therefore \cot\theta = \frac{3 - \cos\alpha}{\sin\alpha}$

CHAPTER IV.

ON THE CENTRE OF GRAVITY.

56. *If any number of parallel forces act at given points in a rigid body, there is a point through which the resultant will always pass, whatever the position in which the body may be placed.* This point is termed the centre of the parallel forces.

Let P_1, P_2, P_3, P_4 be parallel forces acting at A, B, C, D points in a rigid body. Join the points A, B, and in AB take m, so that $Am : mB :: P_2 : P_1$, then, by Art. 32, the resultant of P_1 and P_2 passes through m. Let X be this resultant, and join the points m, C; in mC take n, so that $mn : n\text{C} :: P_3 : X$, then as before the resultant of X and P_3, that is of P_1, P_2, and P_3 passes through n.



Let Y be this resultant, and join n, D: in nD take o, so that $no : o\text{D} :: P_4 : Y$, then the resultant of Y and P_4, that is of P_1, P_2, P_3, and P_4, passes through o.

If now we suppose the body to be moved into a new position, or what would be the same thing, if we suppose the direction of the several forces to be changed, and to assume the position represented in the figure by p_1, p_2, p_3, and p_4, then, since the magnitudes

of the forces are unchanged, the resultant of p_1 and p_2 will as before pass through m, that of p_1, p_2, and p_3, through n, and that of p_1, p_2, p_3, and p_4, through o. The same will take place in any other position of the body, and hence in all positions of the body the resultant of the several forces will pass through o.

57. *Gravity* is the force which attracts every particle of matter towards the centre of the earth. The *weight* of a body is the total force with which that body is drawn towards the earth's centre; it is therefore the same as the resultant of the forces which, in consequence of the existence of gravity, act upon its several particles. These forces may, without sensible error, be regarded as parallel forces. The centre of these parallel forces is called the *centre of gravity*.

The centre of gravity of a body is, then, the point through which will pass, in every position of the body, the resultant of all the forces which, in consequence of gravity, act upon its particles.

58. It follows from the definition, that if the centre of gravity of a body be fixed, the body will rest in any position. For in every position the resultant of all the forces arising from gravity passes through the fixed point, and being met by the resistance of that point, equilibrium is preserved.

59. Since the resultant may be always substituted in place of its component forces, it follows that in considering the influence of any weighty body in producing equilibrium, we may substitute a force equal to the weight of the body, and acting at its centre of gravity.

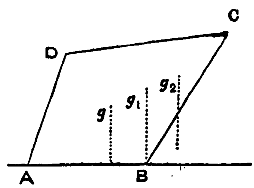
60. *If a body suspended from any point be at rest, the centre of gravity must lie in the vertical line drawn through this point.* For there are two forces in equilibrium; viz., the weight of the body which acts vertically, and at the centre of gravity, and the re-

action of the fixed point. By Art. 11, these forces must lie in the same straight line, and therefore the vertical line through the centre of gravity must pass through the fixed point.

Hence, if a body be suspended successively from two different points, and the vertical lines passing through those points be drawn, the intersection of these two lines will be the centre of gravity.

61. *A body placed with its base upon a plane surface will stand or fall, according as the vertical line through its centre of gravity falls within or without the base.*

Let ABCD be any body, whose base AB rests upon a plane surface. Let the centre of gravity be at g, so that the vertical line through g falls within AB. The weight of the body acting in this line is met by the resistance of the plane, and, consequently, will not cause the body to turn about either A or B.



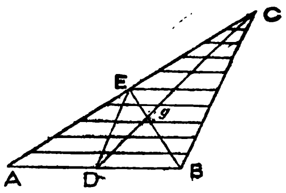
But if the centre of gravity be at g_2, so that the vertical line drawn through the centre of gravity falls without the base, the weight of the body, acting in this line, is not met by the re-action of the plane. There cannot, therefore, be equilibrium, but the body will turn over on to the side BC.

If the centre of gravity be at g_1, so that the vertical line drawn through it pass through one of the extremities of the base, the body will rest; for the weight acting in the vertical line is met by the resistance of the plane: a very slight disturbance will, however, cause the body to fall over.

62. The centre of any perfectly symmetrical figure must also be the centre of gravity; for there can be no reason why the centre of gravity should fall upon one side of that point more than

upon another. Hence, the centre of gravity of a line is its point of bisection; of a parallelogram, the intersection of its diagonals; of a parallelopiped, the intersection of its diagonals; of a circle and sphere, their centres.

63. *To find the centre of gravity of a triangle.* Let ABC be the given triangle. Bisect AB in D and AC in E. Draw CD, BE. If we suppose the triangle to be made up of an infinite number of lines parallel to AB, these lines will be bisected by CD. Consequently the centre of gravity of each of these lines will be in the line CD, and therefore the centre of gravity of the triangle will lie in the line CD. For similar reasons the centre of gravity must lie in the line BE. Therefore g, the intersection of these two lines, must be the centre of gravity required.



Join the points D, E. DE will be parallel to CB, and the triangle ADE similar to the triangle ABC. Hence,

$$DE : BC :: AD : AB;$$

$$\therefore \quad DE = \tfrac{1}{2}BC.$$

But, by similarity of triangles, $gD : gC :: DE : BC$;

$$\therefore \quad gD = \tfrac{1}{2}gC,$$

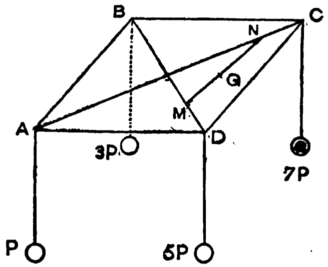
$$\therefore \quad gD = \tfrac{1}{3}DC.$$

Hence, the centre of gravity of a triangle lies in the line joining any vertex with the bisection of the opposite side, at a distance from the side of one-third of this line.

64. *To find the centre of gravity of four weights,* P, 3P, 7P, *and* 5P, *suspended from the angular points of a square weightless plate.*

Let ABCD be the square plate, and let the weights P, 3P, 7P, and 5P be suspended severally from the points A, B, C, D. Draw the diagonals, AC, BD. Take N, so that $CN = \tfrac{1}{8}AC$, and M so that $DM = \tfrac{3}{8}BD$. Then $AN = \tfrac{7}{8}AC$, and $BM = \tfrac{5}{8}BD$.

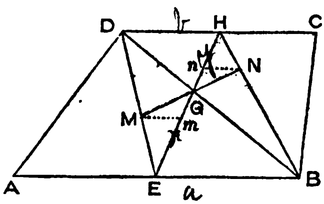
The resultant of P at A, and 7P at C, is a force 8P at N, since $7P \times CN = P \times AN$. And the resultant of 3P at B and 5P at D is also a force 8P at M; since $5P \times DM = 3P \times BM$. We have then, in the place of the original forces, two equal forces acting at M and N. Bisect MN at G, then the resultant of these equal forces will pass through G, and G is the centre of gravity required.



65. *To find the centre of gravity of a trapezium.*

Let ABCD be the trapezium, in which the sides AB, CD are parallel.

Bisect AB in E, and DC in H, and join HE. If the trapezium be supposed to be made up of an infinite number of lines parallel to AB, each of these lines will be bisected by EH, and consequently the centre of gravity of the trapezium must be in the line EH.



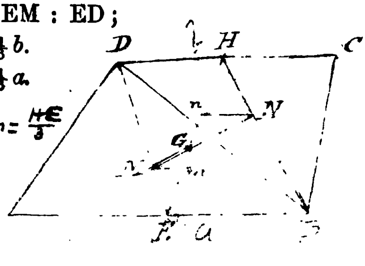
Join DE, and take $EM = \frac{1}{3} ED$. Similarly join HB, and take $HN = \frac{1}{3} HB$. Then M is the centre of gravity of the triangle ABD, and N that of the triangle BCD. Consequently the centre of gravity of the whole figure must be in the line MN; but it also lies in the line EH; therefore the intersection of these lines, or the point G, is the centre of gravity required.

Let GE be called x, and GH y; and let $AB = a$, and $DC = b$. Draw Mm, Nn parallel to AB or CD. Then, since the triangles EMm, EDH are similar,

$$\therefore \quad Mm : DH :: EM : ED;$$

$$\therefore \quad Mm = \tfrac{1}{3} DH = \tfrac{1}{6} b.$$

Similarly, $\quad Nn = \frac{1}{3} EB = \frac{1}{6} a.$



$Mm = \frac{1}{6} b, \; Nn = \frac{1}{6} a; \; Gm = Hn = mn = \frac{HE}{3}$

$\therefore \frac{a}{b} = \frac{Nn}{Mm} = \frac{Gn}{Gm}$

$\therefore \frac{a+2b}{2a+b} = \frac{Gn+2Gm}{2Gn+Gm} = \frac{Gm+\frac{HE}{3}}{Gn+\frac{HE}{3}}$

$= x.$

Again, because the triangles MGm, NGn are similar,

$$\mathrm{G}m : \mathrm{G}n :: \mathrm{M}m : \mathrm{N}n;$$

$$\therefore \quad \mathrm{GE} - \mathrm{E}m : \mathrm{G}h - \mathrm{H}n :: \mathrm{M}m : \mathrm{N}n.$$

But Em and Hn are each equal to $\frac{1}{3}$ EH, or $\frac{1}{3}(x+y)$;

$$\therefore \quad x - \tfrac{1}{3}(x+y) : y - \tfrac{1}{3}(x+y) :: \tfrac{1}{6}b : \tfrac{1}{6}a;$$

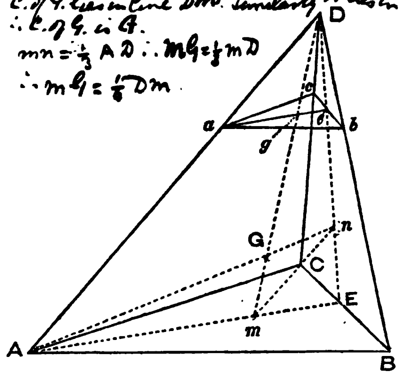
$$\therefore \quad 2x - y : 2y - x :: b : a;$$

$$\therefore \quad x : y :: a + 2b : 2a + b.$$

66. *To find the centre of gravity of a triangular pyramid.*

Let ABCD be any triangular pyramid. Bisect BC in E. Draw AE and DE. Take Em = $\frac{1}{3}$ AE, and En = $\frac{1}{3}$ DE. Join Dm and An. These lines will intersect in a point G, which is the centre of gravity of the pyramid.

c. of G. lies in line Dm. Similarly it lies in An
∴ c. of G. is G.
mn = ⅓ AD ∴ mG = ⅓ mD
∴ mG = ¼ Dm



The pyramid may be conceived to be made up of an infinite number of planes parallel to ABC. Let abc be such a plane. Then, since the parallel planes abc, ABC are intersected by a third plane BCD, the intersections cb, CB are parallel. And since cb is parallel to CB, cb is bisected in e by the line DE; for, by similarity of triangles,

$$ce : \mathrm{CE} :: \mathrm{D}c : \mathrm{DC},$$

and $$cb : \mathrm{CB} :: \mathrm{D}c : \mathrm{DC};$$

$$\therefore \quad ce : cb :: \mathrm{CE} : \mathrm{CB};$$

but $$\mathrm{CE} = \tfrac{1}{2}\mathrm{CB};$$

$$\therefore \quad ce = \tfrac{1}{2} cb.$$

The lines ae, AE, are parallel, for they lie in the same plane

AED, and in the parallel planes *abc*, ABC. Then, by similarity of triangles,

$$eg : Em :: De : DE,$$

and $$ea : EA :: De : DE;$$

$\therefore$ $$eg : ea :: Em : EA;$$

but $$Em = \tfrac{1}{3} EA;$$

$\therefore$ $$eg = \tfrac{1}{3} ea;$$

therefore g is the centre of gravity of the triangle *abc*. In like manner, it may be shown that the centre of gravity of every plane parallel to ABC is in the line Dm. The centre of gravity of the whole pyramid must therefore be in the line Dm. For similar reasons it must also be in the line An. Therefore G, the intersection of these two lines, must be the centre of gravity of the pyramid.

Join mn; then, since Am : mE : : Dn : nE, mn is parallel to AD, and therefore

$$mn : AD :: mE : AE;$$

but $$mE = \tfrac{1}{3}AE;$$

$\therefore$ $$mn = \tfrac{1}{3}AD.$$

The triangles AGD, mGn, are similar; therefore,

$$mG : GD :: mn : AD;$$

$\therefore$ $$mG = \tfrac{1}{3}GD;$$

$\therefore$ $$mG = \tfrac{1}{4}Dm.$$

Therefore the centre of gravity of the pyramid is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

67. *To find the centre of gravity of a pyramid whose base is any polygon.*

Divide the polygon into triangles; and if planes be supposed to pass through the sides of these triangles and the vertex, the pyramid will be divided into triangular pyramids.

Let a plane be drawn parallel to the base, at a distance equal to one-fourth of the altitude of the pyramid, then (Art. 66) the

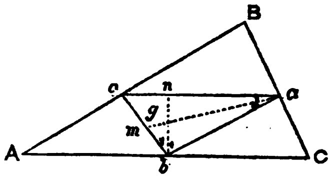
centre of gravity of each of the triangular pyramids will be in this plane.

Again, join the vertex with the centre of gravity of the base. The centre of gravity of every section parallel to the base will be in this line. Hence, the centre of gravity of the whole pyramid will be in this line.

Therefore the centre of gravity is in the line joining the vertex with the centre of gravity of the base, at a point whose distance from the base is one-fourth of this line.

Undone 68. *To find the centre of gravity of the perimeter of a triangle.*

Let ABC be the given triangle. Join *abc*, the points of bisection of the three sides. Draw *am*, *bn*, bisecting the angles *cab*, *abc* respectively. The point *g*, the intersection of these two lines, will be the centre of gravity of the perimeter of the triangle ABC. For the three lines, AB, BC, CA, may be conceived to act at the points *c*, *a*, *b*, with forces represented by their several lengths. The triangles *abc*, ABC are similar; and, therefore,



$$AB : BC : CA :: ab : bc : ca.$$

Hence, the force at *c* : the force at *b* :: *ab* : *ca*. But, since *am* bisects the angle *bac*,

$$bm : mc :: ab : ac;$$

$$\therefore \quad \text{force at } c : \text{force at } b :: bm : mc.$$

Consequently the resultant of the forces at *b* and *c* passes through *m*, and may be represented in value by $ab + ac$. The resultant of this force and the force at *a*, represented by *bc*, must pass through *g*; for

$$gm : ga :: mb : ab;$$

$$\text{but,} \quad mb : ab :: bc : ab + ac;$$

$$\therefore \quad gm : ga :: bc : ab + ac.$$

$\frac{f_c}{f_b} = \frac{ab}{ac} = \frac{bm}{mc}$ ∴ f_{b+c} passes through m, and f_{b+c} may be represented by $(ab + ac)$.

$\frac{f_m}{f_a} = \frac{mb + mc}{ab + ac} = \frac{f_a}{f_{b+c}}$ ∴ f_{a+b+c} passes through g.

The point g is also the cèntre of the circle inscribed in abc. Hence the centre of gravity of the perimeter of a triangle is the centre of the circle inscribed in the triangle formed by joining the bisections of its sides.

69. If the magnitudes and centres of gravity of the several parts of a body of uniform density are known, the centre of gravity of the whole body can be found by the proposition given in Art. 53.

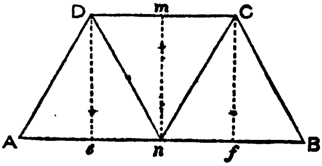
For let M be the total magnitude of the body, and M_1, M_2, M_3, &c. the magnitudes of its several parts. But instead of these bodies, we may suppose forces proportional to their weights to act at their respective centres of gravity. Since the density of the body is uniform, these weights will be proportional to the magnitudes. Consequently, if h be the distance of the centre of gravity of the whole body from any plane, and h_1, h_2, h_3, &c. the known distances of the centres of gravity of the several parts from the same plane;

$$Mh = M_1 h_1 + M_2 h_2 + M_3 h_3 + \&c.$$

$$\therefore \quad h = \frac{M_1 h_1 + M_2 h_2 + M_3 h_3 + \&c.}{M}. \quad \text{or} \quad \bar{h} = \frac{\Sigma(Ph)}{\Sigma(P)}$$

Ex. 1. To find the centre of gravity of half a regular hexagon.

Let ABCD be half a regular hexagon. Bisect AB in n, and join Dn, Cn. Then the given figure is divided into three equilateral triangles, whose centres of gravity are known.



Bisect DC in m, and join mn. Then the centre of gravity must evidently be in this line, and will consequently be found, if we can determine its distance from AB.

Let $De = 3c$, and let the area of each of the equilateral triangles be a. Then the area of the whole figure is $3a$. The distance of the centre of gravity of the triangles ADn, BCn, from AB; or, which is the same thing, its distance from a plane, drawn through AB perpendicular to the plane of the paper, is c. The distance of the centre of gravity of the triangle CDn from AB is $2c$. Then if h denote the distance of the centre of gravity of the whole figure from AB,

$$h = \frac{ac + 2ac + ac}{3a} = \frac{4c}{3}.$$

Ex. 2. Let the upper face of a cube be the base of a square pyramid, to find the centre of gravity of the whole figure, the density being uniform.

Let a = the edge of the cube, and b = the height of the pyramid. The centre of gravity must evidently lie in the line joining the vertex of the pyramid with the centre of the base of the cube. Let the base of the cube be the plane from which the distances of the centres of gravity are reckoned.

The content of the cube is a^3, and the distance of its centre of gravity from the base is $\frac{1}{2}a$.

The content of the pyramid is $\frac{1}{3}a^2b$, and the distance of its centre of gravity from the base of the figure is $a + \frac{1}{4}b$. Therefore, if h denote the distance of the centre of gravity of the whole figure from the same plane,

$$h = \frac{a^3(\frac{1}{2}a) + \frac{1}{3}a^2b(a + \frac{1}{4}b)}{a^3 + \frac{1}{3}a^2b},$$

$$= \frac{6a^2 + 4ab + b^2}{12a + 4b}.$$

70. In like manner the centre of gravity of any part can be found, when the centres of gravity of the whole and the remaining parts are known.

Ex. The exterior of a cup is a cylinder, and its interior is a cone; required the depth of the centre of gravity.

Let A be the area of the circle formed by the rim of the cup, a the height of the cup, and b the depth of the vertex of the cone; then the content of the cylinder is Aa, and the depths of its centre of gravity is $\frac{1}{2}a$.

Also, the content of the cone is $\frac{1}{3}Ab$, and the depth of its centre of gravity is $\frac{1}{4}b$.

Therefore the content of the cup is $A(a-\frac{1}{3}b)$; and if x be the depth of its centre of gravity,

$$\frac{Aa^2}{2}=\frac{Ab^2}{12}+A(a-\tfrac{1}{3}b)x;$$

$$\therefore \qquad x=\frac{6a^2-b^2}{12a-4b}.$$

EXAMPLES.

1. Four heavy particles, whose weights are 1, 3, 5, and 7 lbs., are placed along a weightless rod, so as to divide its length into three equal parts; required the distance of the centre of gravity from the middle point of the rod.

Let a be the length of the rod, then the distance required will be

$$\frac{5a}{24}.$$

2. Three heavy particles, whose weights are P, Q, and R, are connected at given distances by a weightless rod; required the distance of the centre of gravity from the weight Q.

Let a be the distance between P and Q, and b the distance between Q and R; then the distance required is

$$\frac{Pa \sim Rb}{P+Q+R}.$$

3. Find the distance of the centre of gravity of four-fifths of a regular pentagon from the centre of the inscribed circle.

Let a be the radius of the inscribed circle, then the distance required is $\frac{1}{6}a$.

4. A bar, of uniform thickness and density, is bent so as to form four sides of a regular pentagon; find the distance of the centre of gravity from the centre of the inscribed circle.

Let a be the radius of the inscribed circle, then the distance required is $\frac{1}{4}a$.

5. Find geometrically the centre of gravity of half a regular octagon.

6. Find geometrically the centre of gravity of a bent lever of uniform thickness and density.

7. If from a square ABCD, whose diagonals intersect in O, the portion AOD be taken away, the distance of the centre of gravity of the remainder from O is equal to $\frac{1}{9}a$ (a being the side of the square).

8. From a cone, whose height is 8 inches, a similar cone, 4 inches in height, is cut off by a plane parallel to the base; what is the distance of the centre of gravity of the frustum from the base?

The distance required is $1\frac{4}{7}$ inch.

9. The height of a conical shell is 14 inches, and the external diameter of the base is double the internal diameter, what is the distance of the centre of gravity from the base?

The distance required is $3\frac{3}{4}$ inches.

10. Upon AB a side of a square, ABCD describe an equilateral triangle, and let VN be the line joining its vertex with the bisection of the side CD; then, if G be the centre of gravity of the whole figure,

$$GN : VN :: 4 - \sqrt{3} : 4 + \sqrt{3}.$$

11. If from any pyramid whose height is a, a pyramid whose height is b be cut off by a plane parallel to the base, shew that the distance of the centre of gravity of the frustum from the base is equal to

$$\frac{a-b}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

12. Let two spheres of the same density, whose radii are a and b respectively, touch one another externally; the distance of their centre of gravity from the centre of the sphere whose radius is a is equal to

$$\frac{b^3}{a^2 - ab + b^2}.$$

13. Particles, whose weights are as 3, 4, 5, and 8, are placed in the corners of a square weightless plate whose side is 28 inches, what is the distance of the centre of gravity from the centre of the plate?

The distance required is $\frac{7}{5}\sqrt{10}$ inches.

CHAPTER V.

ON THE SIMPLE MACHINES.

71. A *machine* is an instrument, by the agency of which one force can resist or overcome another force not immediately opposite to it in direction.

72. The simplest of such instruments are sometimes termed the *Mechanical Powers*, but are more fittingly named as the *Simple Machines*. These are, the lever, the wheel and axle, toothed wheels, the pulley, the inclined plane, the wedge, and the screw. These differ from each other more in their structure than in the principles of their operation; for when in equilibrium, the wheel and axle, the toothed wheel and the pulley, may be reduced to the lever; and the wedge and the screw are but modifications of the inclined plane.

The two forces which act upon either of these simple machines are, for the sake of distinction, called the *power* and the *weight*, the latter always denoting the force to be resisted or overcome.

73. By the *mechanical advantage* of any machine is meant the ratio of the weight to the power, when in equilibrium; thus, if a power of 2 lbs. sustain a weight of 30 lbs. the mechanical advantage is $30 \div 2$, or 15.

74. THE LEVER. The lever is an inflexible bar, capable of free motion about a fixed axis, called the fulcrum. Unless the contrary be stated, the lever is usually supposed to be without weight.

Levers are of three kinds, according to the relative position of the power, weight, and fulcrum.

When the fulcrum is between the power and the weight, the lever is of the first kind.

When the weight is between the fulcrum and the power, the lever is of the second kind.

When the power is between the fulcrum and the weight, the lever is of the third kind.

The beam of a balance is a lever of the first kind, an oar is a lever of the second kind, and the treadle of a lathe is a lever of the third kind.

Scissors are double levers of the first kind, nut-crackers are double levers of the second kind, and spring shears are double levers of the third kind.

75. *To find the condition of equilibrium in the lever, when the power and the weight act in parallel directions.*

It is evident that there will be equilibrium, if the resultant of the two forces passes through the fulcrum. Let P represent the power, and W the weight; let a be the distance from the fulcrum at which P acts, and b the distance at which W acts. Then, since P and W are by hypothesis parallel forces, their resultant will pass through the fulcrum, if

$$Pa = Wb,$$

whether the fulcrum be between the forces, as in levers of the first kind, or without them, as in levers of the second and third kinds. Hence, when the power and the weight are parallel forces, the condition of equilibrium in a lever of any kind is, that the *power* × *its distance from the fulcrum* = *the weight* × *its distance from the fulcrum;* and, therefore, the mechanical advantage

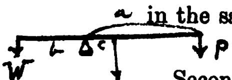
$$= \frac{\text{distance of power from fulcrum}}{\text{distance of weight from fulcrum}}.$$

76. *To find the condition of equilibrium in a lever acted upon by parallel forces, when the weight of the lever itself is regarded.*

Let P denote the power, and W the weight. Let a = the length of the arm at which P acts, and b = the length of the arm at which W acts. Let w = the weight of the lever, and c = the distance of its centre of gravity from the fulcrum.

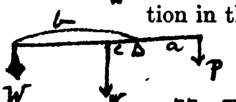
The weight of the lever may be regarded as a force w acting at the centre of gravity. By Art. 47, there will be equilibrium, if the algebraic sum of the moments about the fulcrum be equal to zero.

First. Let the weight of the lever tend to produce revolution in the same direction as P, then



$$Pa + wc - Wb = 0.$$

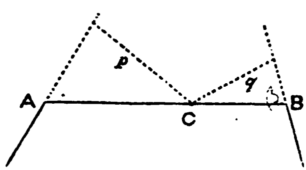
Secondly. Let the weight of the lever tend to produce revolution in the same direction as W, then



$$Pa - wc - Wb = 0.$$

77. *To determine the condition of equilibrium in a lever, when the power and weight act in any direction whatever.*

Let P and Q be forces acting in any direction at the extremities A and B of any lever, which may be either straight, as in the figure, or bent in any manner. From the fulcrum C let fall perpendiculars upon the direction of the forces, and let their lengths be p and q respectively. Then (Art. 30) the resultant of P and Q passes through C, if



$$Pp = Qq,$$

which is, consequently, the condition of equilibrium required.

Cor. If a and b be the arms of any lever, straight or bent, and if P and Q make angles α and β respectively with the arms; then $p = a \sin \alpha$, and $q = b \sin \beta$; consequently, the condition of equilibrium is

$$Pa \sin \alpha = Qb \sin \beta.$$

78. *To determine the pressure upon the fulcrum of a lever.*

The pressure upon the fulcrum is in every case the resultant of the forces acting upon the lever. When these are parallel and act concurrently, the resultant equals their sum; if not concurrent, the resultant equals their difference. Hence, the pressure upon the fulcrum,

in levers of the first kind, is $P+W$,
in levers of the second ...... $W-P$,
in levers of the third ...... $P-W$.

If the weight of the lever itself be regarded, it must be added to or subtracted from the power, according as it acts concurrently or non-concurrently with the power. Then, if $w =$ the weight of the lever, the pressure upon the fulcrum,

in levers of the first kind, is $P+W \pm w$,
in levers of the second, is ... $W-(P \pm w)$,
in levers of the third, is ... $(P \pm w)-W$.

79. An ordinary balance is, mechanically considered, simply a lever of the first kind, provided with pans suspended from its extremities. When the arms of the balance are equal, a body placed in one of the pans will be balanced by an equal weight in the other. When, however, the arms are unequal, the weight in one pan is not equal to that of the body in the other.

80. *To determine the true weight of a body by a false balance.*

Let w be the true weight of the body, and let x and y be the unknown arms of the balance. When w is placed in one pan, let it be balanced by a weight a; and when placed in the other, by a weight b. Then, since w and a, acting at the arms x and y, were in equilibrium,

$$wx = ay.$$

And since w and b, acting at the arms y and x, were in equilibrium,

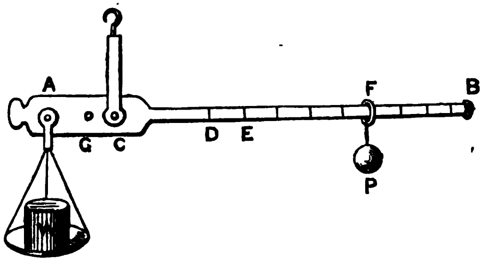
$$wy = bx,$$

$$\therefore \quad w^2xy = abxy,$$

$$\text{or,} \quad w^2 = ab, \text{ or } w = \sqrt{ab}$$

That is, the true weight is a mean proportional between the two false weights.

81. THE STEELYARD. The common steelyard is a lever of the first kind, with very unequal arms, provided with a moveable weight P, which may be suspended from any point in the longer arm. The longer arm is graduated, and the weight of any body suspended from the shorter arm is indicated by the point at which P must be placed in order to balance it.



82. *To graduate the common steelyard.*

First. When the centre of gravity of the instrument (including both the beam and the scale-pan) is in the shorter arm.

Let G be the centre of gravity, let D be the point at which P must be placed, in order to balance the instrument only, and E the point at which P balances the instrument, when a weight of 1 lb. is placed in the scale-pan; then, if successive distances, each equal to DE, be marked along the arm, they will denote the positions of P, corresponding to successive additions of one pound to the weight.

For, let w be the weight of the instrument; then, since the instrument is balanced by P at D,

$$P \,.\, CD = w \,.\, GC \quad \ldots\ldots\ldots\ldots\ldots (\text{i.})$$

Again, since a weight of 1 lb. in the scale-pan is balanced by P at E,

$$P \,.\, (CD + DE) = w \,.\, GC + AC\,;$$

whence, by subtraction,

$$P \,.\, DE = AC \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{ii.})$$

Let a weight of n lbs. in the scale-pan be balanced by P at F, then

$$P\,(CD + DF) = w \,.\, GC + n \,.\, AC\,;$$

First. $w.GC = P.CD.$
$AC + w.GC = P.CD + P.DE.$
$\therefore AC = P.DE.$
$n.AC + w.GC = P.CD + P.DF$
$\therefore n.DE = DF.$

Secondly. $P.CD + w.GC = p.AC$
$P.DE + P.CD + w.GC = (p+1)\,AC$
$\therefore P.DE = AC.$
$P.DF + P.CD + w.GC = (p+n)\,AC$
$DF = n.DE$

whence, by equations (i.) and (ii.),

$$P(CD + DF) = P.CD + nP.DE;$$

and, therefore, $\quad DF = n.DE;$

that is, if n be 2, $DF = 2DE$; if n be 3, $DF = 3DE$, and so on.

Secondly; when the centre of gravity of the instrument is in the larger arm.

Let G be the centre of gravity, falling now to the right of C.

In this case the steelyard will serve to weigh those bodies only whose weight is greater than $\frac{w.GC}{AC}$. Let p be some whole number of pounds greater than $\frac{w.GC}{AC}$, and let p lbs. in the scale-pan be balanced by P at D, and let $(p+1)$ lbs. be balanced by P at E; then, as before, successive distances, each equal to DE, will denote successive increments of 1 lb. in the weight. For, by hypothesis,

$$P.CD + w.GC = p.AC \text{ ....(i.)}$$

and $\quad P.CE + w.GC = (p+1)AC;$

$$\therefore \quad P.DE = AC \text{ ....(ii.)}$$

Let a weight of $(p+n)$ lbs. be balanced by P at F, then

$$P(CD + DF) + w.GC = (p+n)AC;$$

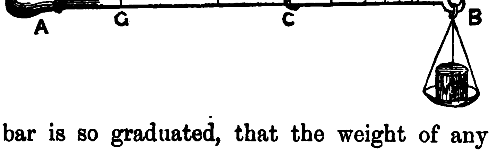
whence, subtracting (i.)

$$P.DF = n.AC;$$

and, therefore, by (ii.)

$$DF = n.DE.$$

83. The Danish Balance. This instrument consists of a straight bar AB, having a heavy knob A at one end, and at the other end B a hook bearing a scale-pan. The fulcrum C is moveable, and the bar is so graduated, that the weight of any body placed in the scale-pan is determined by the position of C.



84. *In the Danish Balance the distances of the fulcrum from the end of the bar, corresponding to successive equal increments of the weight, form a harmonic series.*

Let w be the weight of the instrument, and G its centre of gravity. Let C be the position of the fulcrum when the weight of the instrument balances a weight of n units placed in the scale-pan. Then

$$nBC = w.GC,$$
$$= w(BG - BC);$$

therefore,

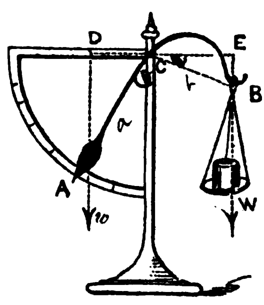
$$BC = \frac{w.BG}{w+n}.$$

Consequently, the successive values of BC, corresponding to the values 0, 1, 2, 3, &c. given to n, are

$$\frac{w.BG}{w}, \frac{w.BG}{w+1}, \frac{w.BG}{w+2}, \frac{w.BG}{w+3}, \text{ \&c.}$$

The reciprocals of these quantities form an arithmetical series, and therefore the quantities themselves are in harmonical progression.

85. THE BENT LEVER BALANCE. This instrument, which is very convenient for determining expeditiously the weight of bodies within a moderate range, consists of a bent lever ACB, one end of which A bears an index, which moves over a graduated quadrant, and the other end B sustains the scale-pan. The thickness of the lever is greatly increased towards A, so that its centre of gravity shall be not far from A. The quadrant may be graduated experimentally, by placing different weights successively in the pan, and marking off the points at which A rests.



The graduation of the quadrant may be also determined by calculation.

Let θ be the angle between the arm AC and the upright stem, corresponding to any weight W. Let w be the weight of the bent lever, and w' the weight of the scale-pan. Then, taking the moments about C,

$$w \cdot CD = (W + w')\, CE.$$

But if a be the length of the arm AC, and b that of the arm BC; and, for greater simplicity, let the angle ACB be a right angle. Then $CD = a \sin\theta$, and $CE = b \cos\theta$. Therefore,

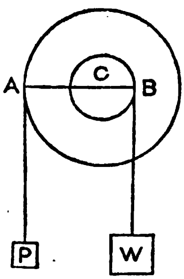
$$w \cdot a \sin\theta = (W + w')\, b \cos\theta,$$

$$\therefore \qquad \tan\theta = \frac{(W + w')\, b}{w\, a}.$$

As θ approaches a right angle, its tangent increases rapidly; and, consequently, a large increase of the weight will then occasion but a small increase in the angle, and hence the balance cannot be depended on for weights above a certain value.

86 THE WHEEL AND AXLE. The wheel and axle consists of a cylinder or axle firmly fixed to a wheel, and having a common axis with it. The weight is attached to a cord passing round the axle, and the power to a cord passing round the wheel.

Let the figure represent a section of such a machine, where C is a point in the common axis, CA the radius of the wheel, and CB the radius of the axle. Let P be the power, and W the weight. These may be regarded as two parallel forces, and if in equilibrium, their resultant must pass through C, and therefore we must have $P \times AC = W \times BC$, or $P \times$ radius of wheel $= W \times$ radius of axle. Hence the mechanical advantage in the wheel and axle is found by dividing the radius of the wheel by the radius of the axle.

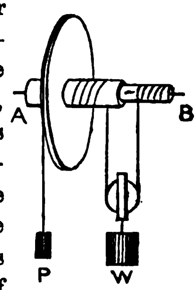


$\frac{W}{P} = \frac{\text{rad. of wheel}}{\text{- - - axle}}$

The wheel and axle admits of various modifications in form. Instead of the wheel, the power may be applied by means of a winch, as in the windlass; or by means of bars inserted into holes

pierced in the axle, as in the capstan. In these cases we have, instead of the wheel, one or more of its radii, and the mechanical advantage is still as before, the radius at which the power acts, divided by the radius at which the weight acts.

87. The advantage in the wheel and axle may be increased either by increasing the radius of the wheel, or by diminishing the radius of the axle. If, however, the wheel be very greatly increased, the machine becomes too unwieldy to be serviceable, and if the axle be much diminished, it becomes too weak to sustain the weight. These difficulties in the way of an indefinite increase of the mechanical advantage are overcome by the simple device of a compound axle, one part of which is of smaller radius than the other. One end of the cord, sustaining the weight, is wound round the thicker part of the axle, and the other end, in a contrary direction, round the thinner part. As the power descends, some part of the cord unwinds from the thinner axle, while another part is wound up around the thicker axle; but as more of the cord is wound up than is let out, the weight is raised by the action of the machine. Let P be the power, W the weight, r the radius of the wheel, a the radius of the thicker axle, and b the radius of the thinner axle. Since the whole weight is supported by the two parts of the cord, the tension in the cord $= \frac{1}{2}$W.



The power and the tension in the cord passing to the thinner axle both act on the one side of the axle, and the tension in the cord passing to the thicker axle acts on the other. By Art. 46, there will be equilibrium, if the sum of the moments of the former about a point in the axis equals the moment of the latter;

$$\therefore \quad Pr + \tfrac{1}{2}Wb = \tfrac{1}{2}Wa,$$

$$\text{or,} \quad Pr = \tfrac{1}{2}W(a-b);$$

$$\therefore \quad P : W :: a-b : 2r.$$

Whence it appears, that the mechanical advantage is equal to

$Pr + \frac{1}{2}Wb = \frac{1}{2}Wa$

$\therefore \frac{W}{P} = \frac{2r}{a-b} = \ldots$

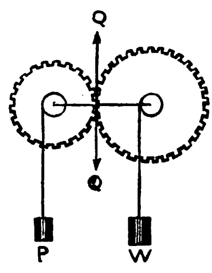
twice the radius of the wheel divided by the difference of the radii of the axles. Part of this advantage is owing to the introduction of the pulley, which, it will be presently seen, doubles the advantage of the machine. Consequently, the advantage of the wheel and axle alone is equal to the radius of the wheel divided by the difference of the radii of the axles, or the machine is equivalent to a simple wheel and axle, having an axle equal to the difference between the thicker and thinner parts of the compound axle.

88. Toothed Wheels. A toothed wheel is a circular plate of wood or metal, having its circumference indented or cut into equal teeth all the way round. If two such wheels, having their teeth of the same magnitude, and at the same distance apart, be so placed that a tooth of one may lie between any two of the other; then, if one of them be turned round by any means, the other will be turned round also.

If the teeth be large, the amount and direction of the pressure which the one wheel exerts upon the other will, unless the teeth be made of a peculiar shape, vary considerably as the wheels revolve; but if the teeth be small in comparison with the size of the wheel, this variation may be disregarded, and the mutual pressure may, without any great error, be treated as constant in magnitude and direction.

89. *To find the ratio of the power and weight in toothed wheels, when the teeth are small.*

Let the power and the weight both act at axles of equal radius, and let c be this radius. Let a be the radius of P's wheel, and b the radius of W's wheel.



Since the teeth are small, the pressure of the one wheel upon the other may be regarded as constant, and as acting in the direction of the common tangent to the two wheels. Let Q denote this pressure. Then, since P and Q are in equilibrium, $Pc = Qa$.

Similarly, $Wc = Qb$;

$\therefore$ multiplying crosswise, $PQbc = WQac$;

$\therefore$ $Pb = Wa$,

or, $P : W :: a : b$.

But since the teeth in each wheel are of the same magnitude and at the same distance apart, the number of teeth in each wheel will be proportional to the circumference, and consequently to the radius. Therefore, P : W : : number of teeth in P's wheel : number of teeth in W's wheel, i.e $\frac{W}{P} = \frac{\text{no. of teeth in W's wheel}}{\text{P's —}}$.

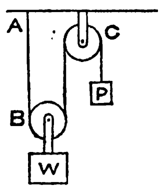
90. The Pulley. A pulley consists of a small wheel, which moves freely about an axis, and allows a cord to pass over any part of its circumference. Unless it be otherwise stated, the wheel is supposed to revolve without friction, and the cord to be perfectly flexible.

With a single fixed pulley, there is neither gain nor loss of power; for, as the tension in every part of the cord is the same, if a weight W be suspended at one extremity, an equal weight must be applied at the other to maintain equilibrium. Hence, in this case, $P = W$.

The effect of a fixed pulley is simply to change the direction of a force.

91. *To find the ratio of the power and weight in a single moveable pulley, when the cords are parallel.*

Let the weight W be attached to the moveable pulley B, and let B be sustained by a cord ABC, one extremity of which is fastened at A, and the other, after passing over the fixed pulley C, sustains the power P.



Let there be equilibrium; then the weight W is sustained by the tension in BA and the tension in BC. But since the cords are parallel, these tensions may be regarded as two parallel forces, and therefore W

must equal their sum. But the tension of the cord is the same throughout, and is equal to P;

$$\therefore \qquad W = 2P, \text{ or } P = \tfrac{1}{2}W.$$

92. The following combinations of pulleys are termed respectively, the first, second, and third system of pulleys.

In the first system of pulleys, each pulley hangs by a separate cord, one end of which is fastened to a fixed beam, and the other to the pulley above it. (See fig. Art. 93.)

In the second system of pulleys, the same cord passes round all the pulleys, which are arranged in two blocks, one of which is fixed, and the other bears the weight. (See fig. Art. 95.)

In the third system of pulleys, each cord is attached to the weight. (See fig. Art. 96.)

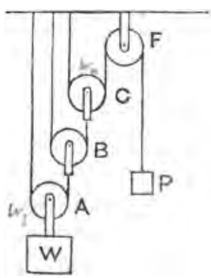
93. *To find the ratio of the power and weight in the first system of pulleys, when the weight of the pulleys is disregarded.*

Let A, B, C be three moveable pulleys, and let P and W be in equilibrium.

By article 91,

$$\text{tension in } AB = \tfrac{1}{2}W,$$
$$\text{tension in } BC = \tfrac{1}{2} \text{ tension in } AB = \tfrac{1}{4}W,$$
$$\text{tension in } CF = \tfrac{1}{2} \text{ tension in } BC = \tfrac{1}{8}W.$$

But, by Art. 90, tension in CF = P, therefore, when there are three moveable pulleys, $P = \frac{1}{8}W$ or $W = 8P$. Similarly, if there be four pulleys, $W = 16P$; if five, $W = 32P$, and so on. Hence, in the first system of pulleys, when the weight of the pulleys is disregarded, the weight is found by doubling the power as many times as there are pulleys, and the power is found by halving the weight as many times as there are pulleys.



Or, if n be the number of moveable pulleys,

$$P = \frac{W}{2^n}, \text{ or } W = 2^n P.$$

wt. of pulleys not regarded, $W = 2^n P$ | First system of Pulleys
being $W - w = 2^n(P - w)$
or $W + w_1 + 2w_2 + 2^2 w_3 + \cdots + 2^{n-1} w_n = 2^n P$.
w_1 is the lowest pulley, w_n is the highest pulley

94. *To find the ratio of the power and weight in the first system of pulleys, when the weight of the pulleys is regarded.*

The total weight acting at each pulley is the tension of the string attached to the block, together with the weight of the block.

The entire weight at A is the weight W, together with the weight of the pulley A; one-half of this sum will give the tension in AB.

The entire weight at B is the tension in AB, together with the weight of the pulley B; one-half of this sum will give the tension in BC.

The entire weight at C is the tension in BC, together with the weight of the pulley C; one-half of this sum will give the tension in CF, which is equal to the power P.

Hence, when W is given to find P: to W add the weight of the lowest pulley, and divide by 2; add the weight of the next pulley, and again divide by 2. Repeat this process as many times as there are moveable pulleys, the result will give P.

If P be given, W is found by the inverse processes. Double P, and subtract the weight of the highest pulley. Double again, and subtract the weight of the next pulley, and so on as many times as there are moveable pulleys.

If n be the number of moveable pulleys, whose weights are severally w_1, w_2, &c...w_n,

$$P = \frac{W + w_1}{2^n} + \frac{w_2}{2^{n-1}} + \text{\&c.} + \frac{w_n}{2},$$

$$= \frac{1}{2^n}(W + w_1 + 2w_2 + \text{\&c.} + 2^{n-1}w_n).$$

Hence, if all the pulleys be of the same weight, w,

$$P = \frac{W + (2^n - 1)w}{2^n}, \text{ or } W = 2^n P - (2^n - 1)w,$$

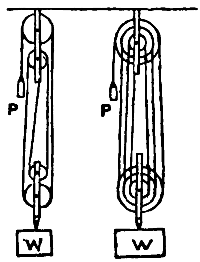
$$\text{and } \therefore \quad W - w = 2^n(P - w).$$

95. *To find the ratio of the power and weight in the second system of pulleys.*

W + w = nP, where n is the no. of portions of the cord in contact with the lower block

Second system of pulleys

Since there is but one cord, and P is attached to one extremity of it, the tension in every part is equal to P. Hence, if n be the number of the portions of this cord in contact with the lower block, the weight supported will be nP; therefore, in this system,



$$W = nP, \text{ or } P = \frac{1}{n}W.$$

If the weight of the lower block be w, the total weight supported is $W + w$; therefore, if the weight of the block be regarded,

$$W + w = nP, \text{ or } P = \frac{1}{n}(W + w).$$

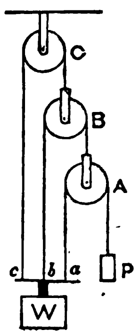
96. *To find the ratio of the power and weight in the third system of pulleys, when the weight of the pulleys is disregarded.*

Let there be three pulleys A, B, C; then

tension in Aa = P,

tension in Bb = pressure on A = 2P,

tension in Cc = pressure on B = 4P.



But as the weight is supported by the tensions in the three cords Aa, Bb, and Cc, and since the cords are parallel, W must equal the sum of the tensions; therefore,

$$W = P + 2P + 4P = 7P.$$

Similarly, if there be n pulleys,

$$W = P + 2P + 2^2P + \&c. + 2^{n-1}P,$$
$$= (2^n - 1)P.$$

97. *To find the ratio of the power and weight in the third system of pulleys, when the weight of the pulleys is regarded.*

The tension in Aa (fig. Art. 96) equals the power. The tension in Bb equals twice the power increased by the weight of the lowest pulley. The tension in Cc equals twice the tension in Bb,

Let the weights of the pulleys beginning with the lowest be $w_1, w_2, w_3, \ldots w_n$ (moveable)
$W = (2^n-1)P + (2^{n-1}-1)w_1 + (2^{n-2}-1)w_2 + \ldots + w_{n-1}$ | Third System of Pulleys.
[illegible] $+ \ldots = (2^n-1)(P+w)$
and $W = (2^n-1)P$

increased by the weight of the second pulley. The weight W will be equal to the sum of the three tensions..

Hence generally, add together as many terms of the following series as there are pulleys; viz., the power, twice the power + the weight of the lowest pulley, twice the preceding + the weight of the next pulley, and so on; this sum will be equal to the weight supported.

If there be n pulleys, whose weights, beginning with the lowest, are severally w_1, w_2, &c... w_n, then

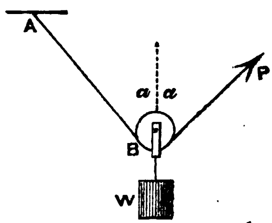
$$W = (2^n - 1)P + (2^{n-1} - 1)w_1 + (2^{n-2} - 1)w_2 + \&c. \ldots + w_{n-1}.$$

Or, if all the moveable pulleys be of the same weight w,

$$W = (2^n - 1)(P + w) - nw,$$

and $\therefore$ $$W + nw = (2^n - 1)(P + w).$$

98. *To find the ratio of the power and weight in the single moveable pulley, when the cords are not parallel.*



Let $2a$ be the angle between the cords. The pulley is at rest under the action of three forces; namely, the weight W and the tension in the two parts of the cord, which tension is equal to P. Then, by Art. 22,

$$P : W :: \sin a : \sin 2a :: 1 : 2 \cos a;$$ or h : P [illegible] + P cos [illegible]

$$\therefore \quad W = 2P \cos a.$$

99. Hence, if two moveable pulleys be combined, as in the first system, with the cords inclined at angles $2a$ and 2β respectively,

$$W = 2^2. P \cos a \cos \beta.$$

For, if T be the tension in the cord bearing the lower pulley, then, by the preceding Article,

$$W = 2T \cos a,$$

and $$T = 2P \cos \beta;$$

therefore, $$W = 2^2. P \cos a \cos \beta.$$

And so generally, if there be n moveable pulleys, and the angles between the cords are $2a_1, 2a_2, 2a_3 \ldots\ldots 2a_n$, respectively; then,

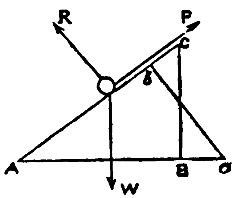
$$W = 2^n \, P \cos a_1, \cos a_2 \ldots\ldots \cos a_n.$$

Hence, if a_1, a_2 &c., be equal to one another,

$$W = P. \, (2 \cos a)^n.$$

100. THE INCLINED PLANE. *To find the ratio of the power and weight in the inclined plane, when the power acts parallel to the plane.*

Let W be a weight resting upon the inclined plane AC, and supported by a power P acting parallel to the plane. The forces in equilibrium are gravity, the power P, and the re-action of the plane acting perpendicularly to the plane. Make $Ab = AB$, and $Ac = AC$, then the triangle Abc is in all respects equal to the triangle ABC. In the triangle Abc,

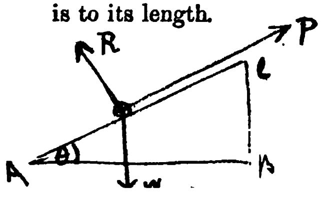


bc is perpendicular to the direction of the power P,
Ac „ „ the direction of gravity,
Ab „ „ the re-action of the plane;
∴ by Article 20, $P : W :: bc : Ac$,
or, $P : W :: BC : AC$.

That is, P is to W as the height of the plane is to its length. The mechanical advantage, or the ratio of W to P is, consequently, the ratio of the length of the plane to the height. In a similar manner the pressure on the plane may be determined. This is equal and opposite to the re-action of the plane. Let R denote this re-action,

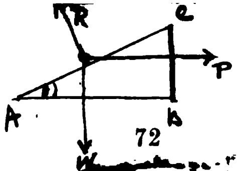
$$R : W :: Ab : Ac,$$
$$:: AB : AC.$$

That is, the pressure on the plane is to W as the base of the plane is to its length.



$P = W \sin\theta \therefore \frac{W}{P} = \frac{AC}{BC} = \frac{l}{h}$

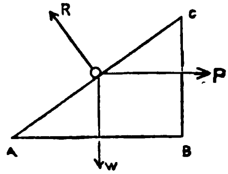
$R = W \cos\theta \therefore \frac{R}{W} = \frac{AB}{AC} =$ [illegible]



$P \cos\theta = W \sin\theta \therefore \frac{W}{P} = \frac{AB}{BC} = \frac{b}{h}$

$R \cos\theta = W \therefore \frac{R}{W} = \frac{AC}{AB} = \frac{l}{b}$

101. *To find the ratio of the power and the weight in the inclined plane, when the power acts horizontally.*



In this case W is kept at rest by three forces; viz., the force of gravity acting vertically, the power acting horizontally, and the re-action of the plane acting perpendicularly to the plane. The sides of the triangles ABC are severally perpendicular to the directions of these forces; viz.—

BC is perpendicular to the direction of P,
AB .... W,
AC .... R.

Therefore P : W : : BC : AB; or the power is to the weight as the height of the plane is to the base.

Hence the mechanical advantage is the ratio of the base to the height.

Also, R : W : : AC : AB; or the pressure on the plane is to the weight as the length of the plane is to the base.

102. *Two weights, connected by a cord which passes over a pulley at the summit, rest upon a double inclined plane; to find their ratio when in equilibrium.*

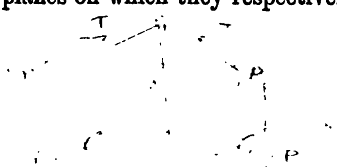
Let l and l' be the length of the two planes supporting respectively the weights W and P. Let h be the common height of the planes. If T denote the tension in the cord, then, since W is supported on an inclined plane by a force T acting along the plane, by Article 100,

$$T : W :: h : l;$$

Similarly, $\qquad T : P :: h : l',$

$\therefore \qquad P : W :: l' : l.$

That is, the two weights are in the same ratio as the lengths of the planes on which they respectively rest.



103. *To find the vertical and horizontal pressures, when a force acts perpendicularly to a given inclined plane.*

Let R in the figure of Art. 101 represent the given force, W and P vertical and horizontal forces in equilibrium with R, and consequently equal and opposite to the vertical and horizontal pressures exerted by R.

But W : R :: AB : AC; therefore,

$$\text{the vertical pressure} = R \times \frac{\text{base of the plane}}{\text{length of the plane}}.$$

And P : R :: BC : AC; therefore,

$$\text{the horizontal pressure} = R \times \frac{\text{height of the plane}}{\text{length of the plane}}.$$

104. *To find the pressure upon an inclined plane in terms of the power and the mechanical advantage.*

Let P be the power and a the mechanical advantage. Let R denote the pressure on the plane, b the base of the plane, h the height, and l the length. Then, if the power acts along the plane, Art. 100,

$$R : P :: b : h :: \sqrt{(l^2 - h^2)} : h;$$

$$R = P\frac{\sqrt{(l^2 - h^2)}}{h},$$

$$= P\sqrt{\left(\frac{l^2}{h^2} - 1\right)};$$

and, therefore, since $a = \frac{l}{h}$

$$R = P\sqrt{(a^2 - 1)}.$$

Again, if the power act horizontally, by Art. 101,

$$R : P :: l : h :: \sqrt{(b^2 + h^2)} : h;$$

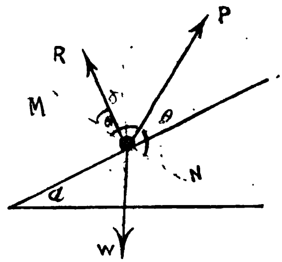
$$R = P\frac{\sqrt{(b^2 + h^2)}}{h},$$

$$= P\sqrt{\left(\frac{b^2}{h^2} + 1\right)};$$

but $a = \frac{b}{h}$, and, therefore,

$$R = P\sqrt{(a^2 + 1)}.$$

105. *To find the ratio of the power and the weight in the inclined plane, when the power acts in any direction whatever, and also to find the pressure on the plane.*



Let the power be inclined to the plane at an angle θ. Then, by the triangle of forces,

$$P : W :: \sin \hat{RW} : \sin \hat{PR}.$$

But $\hat{RW} = 180^\circ - \alpha$, and $\hat{PR} = 90^\circ - \theta$; therefore,

$$P : W :: \sin \alpha : \cos \theta,$$

or, $$W = \frac{P \cdot \cos \theta}{\sin \alpha}.$$

$P \cos \theta = W \sin \alpha$ $\therefore W = \frac{P \cos \theta}{\sin \alpha}.$

To find the pressure on the plane, we have

$$R : W :: \sin \hat{PW} : \sin \hat{PR},$$

but $\hat{PW} = 90^\circ + \alpha + \theta$; therefore,

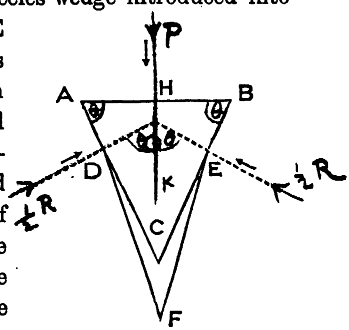
$$R : W :: \cos (\alpha + \theta) : \cos \theta,$$

or, $$R = \frac{W \cdot \cos (\alpha + \theta)}{\cos \theta}.$$

Also $\hat{PW} = 90 + \alpha + \theta$ $\therefore \hat{NW} = \alpha + \theta$ $\therefore R \cos \theta = W \cos \overline{\alpha + \theta}$ $\therefore R = \frac{W . \cos(\alpha + \theta)}{\cos \theta}.$

106 THE WEDGE. *To find the ratio of the power and the resistance in an isosceles wedge.*

Let ABC be the section of an isosceles wedge introduced into the cleft DFE, and let the points DE be similarly situated on the two sides of the wedge. The resistance on each side of the wedge will be the same, and if R = the total resistance, the resistances at D and E will each $= \frac{1}{2}$R; and they act perpendicularly to the sides of the wedge. Let a power P act at the point H, the centre of the back of the wedge. The directions of these three forces when produced will meet in a point G; they may, therefore, be considered as three forces acting



$P = \frac{1}{2} R \cos \theta + \frac{1}{2} R \cos \theta$

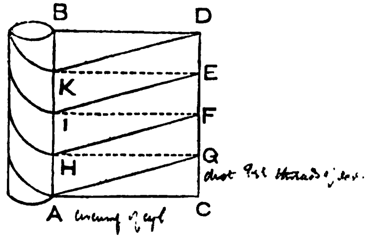
$\therefore \frac{P}{R} = \cos \theta = \frac{\frac{1}{2} AB}{AC}$

upon a point, and in equilibrium. The sides of the triangle ABC are severally perpendicular to the directions of the three forces, and therefore,

$$P : \tfrac{1}{2}R :: AB : AC;$$
$$\therefore \quad P : R :: \tfrac{1}{2}AB : AC.$$

That is, the power is to the total resistance, as half the back of the wedge is to the side of the wedge.

107. The Screw. Let AB be a cylinder, and AD a rectangle, whose base AC is equal to the circumference of the cylinder. Let the side CD be divided into any number of equal parts. Join AG, and through the points F, E, D, draw FH, EI, DK parallel respectively to AG. Then, if the rectangle AD be wrapped round the cylinder, the parallel lines AG, HF, &c. will trace out the continued spiral line called the screw.



Hence, any resistance to be overcome by a screw may be regarded as a weight resting upon an inclined plane, whose base is equal to the circumference of the cylinder, and whose height is equal to the distance between the threads of the screw.

When the screw is used for mechanical purposes, the power is always applied in a direction perpendicular to the axis of the cylinder, and consequently parallel to the base of the inclined plane forming the screw line; and, therefore,

$$P : W :: \text{height of inclined plane} : \text{the base};$$

that is, P : W :: distance between the threads of the screw : circumference of the cylinder.

EXAMPLES.

i. *Problems not requiring a knowledge of Trigonometry.*

1. A bar of uniform thickness and density, whose weight is 6 lbs., and length 4 ft. 2 in., is used as a lever of the first kind; the fulcrum is 10 inches from one end of the bar; what power will sustain a weight of 25 lbs.?

The power required is 4 lbs.

2. A bar of uniform thickness and density, weighing 2 lbs. per foot, is used as a lever of the first kind; the fulcrum is 9 inches from one end of the bar; what must be the length of the bar when its weight balances a weight of 36 lbs. suspended at the end of the shorter arm?

The length of the bar is 6 feet.

3. In a weightless lever of the first kind, a power P acting at an arm a, sustains the same weight as the power Q acting at an arm b; required the length of the lever, and the weight sustained.

The length of the lever is equal to

$$\frac{ab\,(\mathrm{P}-\mathrm{Q})}{\mathrm{P}a-\mathrm{Q}b},$$

and the weight sustained is equal to

$$\frac{\mathrm{P}a-\mathrm{Q}b}{b-a}.$$

4. Two men, able to exert forces of 260 and 300 lbs. respectively, work the handle of a winch and axle. The radius of the axle is 5 inches, what must be the length of the arm of the winch that the men may be just able to raise a weight of 2 tons.

The required length is 3 ft. 4 in.

5. A weight of 10 lbs. is suspended from a single moveable pulley, the cords are inclined at an angle of $30°$; required the tension in the cord.

The required tension is $5\,(\sqrt{6}-\sqrt{2})$ lbs.

6. Five pulleys, each weighing 3 lbs., are arranged according to the first system; what weight will be supported by a power of 20 lbs.?

The required weight is 547 lbs.

7. In the first system of pulleys, if p be the power necessary to balance the weight of the blocks, and if a be the mechanical advantage of the system when the weight of the blocks is disregarded; then will,

$$W = a\,(P - p).$$

8. In the first system of pulleys, composed of blocks of equal weight, show that the power required to balance the weight of the blocks approaches more and more nearly the weight of one of them, as the number of blocks is increased.

9. A weight $2P$ is supported on an inclined plane by two equal forces P and P, one acting horizontally and the other along the plane; required the inclination of the plane.

The plane rises 4 in 5.

10. A weight $2P$ is supported on an inclined plane by two equal forces P and P, one acting horizontally and the other making with the plane an angle equal to the inclination of the plane; required the inclination of the plane and the pressure upon the plane.

The inclination of the plane is $45°$, and the pressure upon the plane is equal to $P\sqrt{2}$.

11. If the arms of a lever of the first kind, of uniform thick-

ness and density, be a and b, and if the weights P and W be in equilibrium, what is the weight of the lever?

The required weight is equal to

$$\frac{2\,(\mathrm{W}b-\mathrm{P}a)}{a-b}.$$

12. If the same bar be used as a lever of the second kind, having as before the shorter arm equal to b, and if the weights P and W be in equilibrium when the weight of the bar acts concurrently with W, what is the weight of the lever?

The required weight is equal to

$$2\left(\mathrm{P}-\frac{\mathrm{W}b}{a+b}\right).$$

13. If P and W be in equilibrium in the simple wheel and axle, and if P be caused to descend, show that P × distance through which P moves = W × distance through which W moves.

14. Show the same in the case of the compound wheel and axle.

15. If two weights P and W, connected by a cord which passes over a pulley at the summit, are in equilibrium upon a double inclined plane, show that, when the weights are put in motion, P × vertical height through which P moves = W × vertical height through which W moves.

16. In a single moveable pulley, if the angle between the cords be 60°, show that $\mathrm{W}=\sqrt{3}\,.\,\mathrm{P}$.

17. If three pulleys, whose weights are w_1, w_2, and w_3, be arranged according to the first system, and P and W be in equilibrium, show that if P descend through a space a, and x_1, x_2, x_3 be the spaces through which the pulleys severally rise in consequence; then,

$$\mathrm{P}a=\mathrm{W}x_1+w_1x_1+w_2x_2+w_3x_3.$$

18. When a weight is supported upon a plane whose inclination is a, what angle must the direction of the power make with the plane that the resistance may be equal to the power?

The required angle is equal to $90° - 2a$.

19. If the inclination of a plane be 30°, and the direction of the power make with the plane an angle of 30°, show that

$$P : W :: 1 : \sqrt{3}.$$

ii. *Examples requiring a knowledge of Trigonometry.*

20. If P and Q, connected by a string passing over a fixed pulley, be in equilibrium on a double inclined plane; if a and β be the inclination of the planes, and θ and ϕ the angles which the two portions of the string make with the planes; then,

$$\frac{P}{Q}=\frac{\cos\theta}{\cos\phi}\cdot\frac{\sin\beta}{\sin a}.$$

21. A weight $(m+n)$ P is supported upon an inclined plane by two forces, one mP acting horizontally, and the other nP making an angle with the plane equal to the inclination of the plane; required the inclination of the plane.

The inclination of the plane is 45°.

22. A weight of 13 lbs. is supported upon an inclined plane by a power of 10 lbs., making an angle with the plane equal to half the inclination of the plane; required the inclination of the plane.

The inclination of the plane is equal to

$$\tan^{-1}\left(\frac{120}{119}\right).$$

23. A cord fastened at A passes under a moveable pulley bearing a weight P, it then passes over a fixed pulley at B, and passing under a second moveable pulley bearing a weight Q, is finally fas-

tened to a fixed peg at C; required the tension in the cord when the angle at one of the moveable pulleys is double the angle at the other.

Let P be greater than Q, then the tension required is equal to

$$\frac{\sqrt{(Q^2+8P^2)}-Q}{4}.$$

24. Find the pressure on the axis of a winch and axle.

Let P be the power, a the mechanical advantage, and θ the angle between the arm of the weight and the arm of the power, then the pressure on the axis is equal to

$$P\sqrt{(a^2-2a\cos\theta+1)}.$$

CHAPTER VI.

ON COMBINATIONS OF THE SIMPLE MACHINES.

108. *To find the mechanical advantage of any combination of machines.*

In a combination of machines, the weight of the first machine is the power of the second, the weight of the second the power of the third, and so on. Let a, b, c, be the separate advantages of three machines in combination. Let P and Q be the power and weight in the first, Q and R those in the second, R and W those in the third; then,

$$Q = aP,$$

and $$R = bQ,$$

and $$W = cR.$$

Multiplying together $$QRW = abc\,PQR;$$

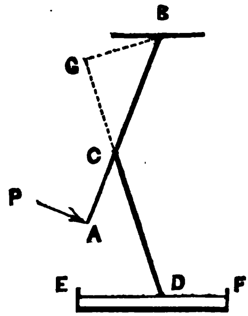
$$\therefore \quad W = abc\,P.$$

Therefore abc represents the mechanical advantage of the combination; that is, the advantage of the combination is equal to the product of the separate advantages of the component machines.

In like manner it may be shown that the mechanical advantage of the combination of any number of machines is equal to the product of the separate advantages of the component machines.

109. THE KNEE LEVER. This is a combination of levers, used in some forms of the printing press, and in other cases where a very great pressure is required to act through a very small space. It consists of a bar ABC, moveable about a fixed pivot at B, and

fastened to a second bar at C. The bar CD is connected by a pivot at D with a moveable plate EF. The lengths CB and CD are so taken that the two bars are nearly in the same straight line, when the plate EF is in its lowest possible position. The power is applied at A, at right angles to AB, and the plate is in consequence pressed down with a very great force.



Let W be the thrust transmitted along the bar CD; then, since CD is nearly perpendicular to the plate when the instrument is in operation, W will very nearly represent the effective pressure on EF.

To determine the ratio of W to P, draw BG at right angles to DC produced. The bar AB is in equilibrium under the action of the force P, and a force equal and opposite to the thrust in CD, that is, a force W acting in the direction CG; wherefore, taking their moments about the fixed point B, we have,

$$\mathrm{W} \times \mathrm{BG} = \mathrm{P} \times \mathrm{AB};$$

$$\therefore \qquad \mathrm{W} = \mathrm{P} \times \frac{\mathrm{AB}}{\mathrm{BG}}.$$

But when the bars are nearly in the same straight line, BG is very small, and consequently $\frac{\mathrm{AB}}{\mathrm{BG}}$ is very large.

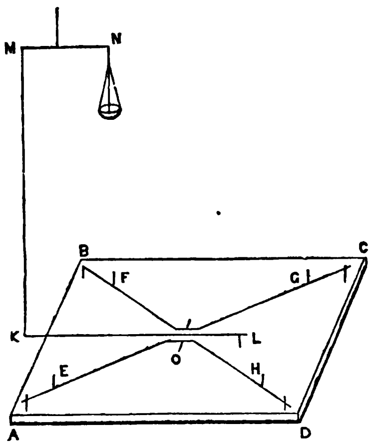
110. If the larger arm of AB be a, and the shorter b, and if, when the instrument is in operation, the angle ACD be θ, then

$$\mathrm{W} = \frac{\mathrm{P}a}{b \sin \theta},$$

and therefore the mechanical advantage is equal to

$$\frac{a}{b \sin \theta}.$$

111. The Weighing Machine. This is a combination of levers, so arranged as to furnish a convenient means for determining the weight of carriages and other heavy bodies. In a rectangular framework ABCD are placed four equal levers of the second order, having their fulcra at A, B, C, and D, and their other extremities connected together at O, and also fastened there by a pivot to a lever KL. This lever is also of the second order, having its fulcrum at L, and is connected by



a rod KM with a small balance MN. At E, F, G, and H, are four pins, severally equidistant from A, B, C, and D. Upon these pins a platform (not represented in the figure) rests, and upon the platform the body to be weighed is placed. When used for weighing carriages, the framework is sunk below the road, so that the platform may be on a level with the road. In place of the balance MN, a small steelyard is frequently used.

In determining the mechanical advantage of this machine, it must be observed, that the four levers, AO, BO, CO, DO, are not in combination, but are simply a contrivance for the convenient support of the weight. The mechanical advantage is simply that of one of them. For let W be the entire weight on the platform, and let V, X, Y, and Z, be the parts of it, acting severally at the points E, F, G, and H, and let m be the advantage of each of the four levers; then,

$$V \text{ at } E \text{ will be balanced by } \frac{V}{m} \text{ at } O,$$

$$
\begin{array}{llll}
X \text{ at } F & \text{will be balanced by} & \dfrac{X}{m} & \text{at } O, \\[2ex]
Y \text{ ,, } G & \text{,,} \quad \text{,,} & \dfrac{Y}{m} & \text{,, } O, \\[2ex]
Z \text{ ,, } H & \text{,,} \quad \text{,,} & \dfrac{Z}{m} & \text{,, } O.
\end{array}
$$

Hence, the entire pressure at O is equal to

$$\frac{V+X+Y+Z}{m};$$

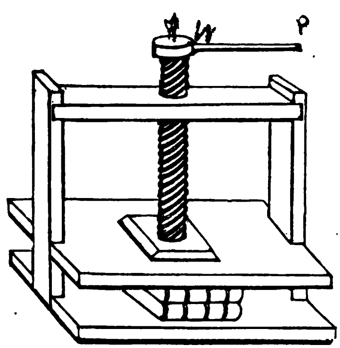
that is, is equal to $\frac{W}{m}$.

Hence, the instrument is equivalent to a combination of only three levers. Let A and a be the lengths of the longer and shorter arms of the lever AO, B and b those of the lever KL, and C and c those of the lever MN; then,

$$\text{mechanical advantage} = \frac{A}{a} \times \frac{B}{b} \times \frac{C}{c},$$

$$= \frac{ABC}{abc}.$$

112. THE SCREW PRESS. This instrument is a combination of the screw and the lever, the power being applied at the extremity of a lever, connected with the screw at right angles with its axis. The fulcrum of the lever is at the axis of the screw, and the pressure exerted by it is at the circumference of the screw. The lever is consequently a lever of the second kind, having its shorter arm equal to the radius of the screw.



Let r be the radius of the screw, d the distance between the

threads, and l the distance of the point at which the power acts from the axis of the screw; then,

$$\text{advantage of lever} = \frac{l}{r},$$

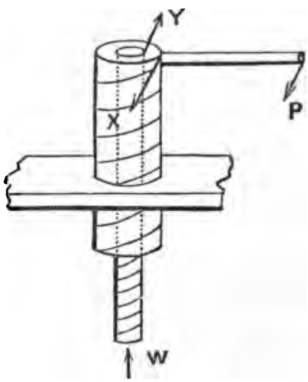
$$\text{advantage of screw} = \frac{2\pi r}{d};$$

therefore,

$$\text{advantage of press} = \frac{2\pi l}{d}.$$

Hence, the advantage of the press is equal to the circumference described by the power divided by the distance between the threads.

113. Hunter's Screw. In the common screw press, the mechanical advantage is increased either by increasing the length of the lever, or by diminishing the distance between the threads. Practically, however, the increase cannot be continued indefinitely by either method. If the length of the lever be too great, the instrument is unwieldy; and if the threads be too fine, they will not be strong enough to sustain the pressure exerted upon them. These difficulties are overcome by the screw invented by Mr. Hunter, and called after his name.



Instead of the single screw of the common screw press, he substituted two screws, one of which worked within the other. The larger screw, as in the common press, works through a nut; and the smaller screw is fastened to the moveable plate. The distance between the threads of the smaller screw is less than that between the threads of the larger screw.

To determine the mechanical advantage of the combination,

let P be the power applied at the end of the lever, and let W be the resistance sustained by it.

Resolve P into two forces X and Y, acting respectively at the circumferences of the two screws; then, if l be the distance of P from the common axis of the screws, r and r' the radii of the screws, by Art 43,

$$Pl = Xr - Yr' \text{.........(i.)}$$

But since a weight W is sustained by a power X acting at the circumference of the larger screw, if d be the distance between the threads,

$$X = \frac{Wd}{2\pi r} \text{...............(ii.)}$$

In like manner, if d′ be the distance between the threads of the smaller screw,

$$Y = \frac{Wd'}{2\pi r'} \text{............(iii.)}$$

Substituting in equation i. the values given by ii. and iii., then

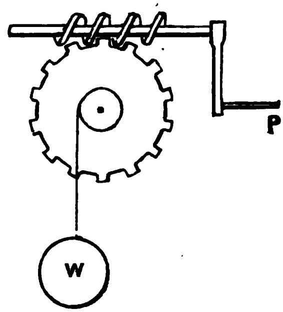
$$Pl = \frac{Wd}{2\pi} - \frac{Wd'}{2\pi};$$

therefore,

$$\frac{W}{P} = \frac{2\pi l}{d-d'};$$

or the mechanical advantage is the circumference described by the power divided by the difference in the intervals between the threads of the larger and smaller screw.

114. The Endless Screw. This instrument is a combination of the screw, and the wheel and axle. The screw is worked by a winch; and the threads of the screw, instead of working in a nut, press against the teeth formed on the circumference of the wheel.



Let l be the arm of the winch, d the distance between the threads of the screw, R the radius of the wheel, and r the radius of the axle; then, by Art. 112,

$$\text{advantage of winch and screw} = \frac{2\pi l}{d},$$

and, by Art. 86,

$$\text{advantage of wheel and axle} = \frac{R}{r},$$

therefore,

$$\text{advantage of endless screw} = \frac{2\pi l}{d} \cdot \frac{R}{r}.$$

Let n be the number of teeth in the circumference of the wheel, then, since the teeth of the wheel are to work in the screw,

$$\frac{2\pi R}{n} = d,$$

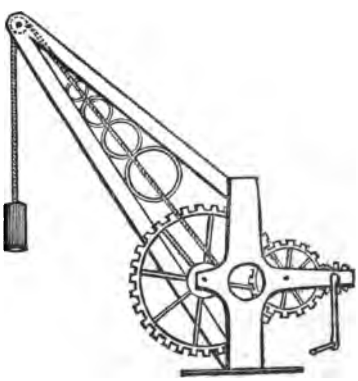
and, therefore,

$$\frac{2\pi R}{d} = n;$$

consequently,

$$\text{advantage of endless screw} = \frac{nl}{r}.$$

115. THE CRANE. This machine, used for raising and lowering heavy weights, is a combination of wheels and axles. A common form is represented in the figure. This consists of a winch and axle, and of two wheels and axles. The axle of the winch, and the axle of the first wheel, are furnished with teeth,* as are also the two wheels. The power is applied at the handle of the winch, and the weight is sustained by a rope or chain passing round the axle of the second wheel.



* A toothed axle is commonly called a *pinion*, and its teeth are termed *leaves*.

Let l be the length of the arm of the winch, and a the radius of its axle, then

$$\text{advantage of winch and axle} = \frac{l}{a}.$$

Let R_1 be the radius of the first, and r_1 the radius of its axle, then

$$\text{advantage of first wheel and axle} = \frac{R_1}{r_1}.$$

Let R_2 be the radius of the second wheel, and r_2 the radius of its axle, then

$$\text{advantage of second wheel and axle} = \frac{R_2}{r_2}.$$

Consequently,

$$\text{advantage of crane} = \frac{l\, R_1\, R_2}{a\, r_1\, r_2}.$$

If N_1, N_2, be the number of teeth respectively in the two wheels, and n_1, n_2, those in the axles of the winch and of the first wheel, then

$$\frac{R_1 . R_2}{a\, r_1} = \frac{N_1\, N_2}{n_1\, n_2},$$

and therefore

$$\text{advantage of crane} = \frac{l . N_1\, N_2}{n_1 . n_2 . r_2}.$$

EXAMPLES.

1. In a screw press, the length of the lever is l; what must be the distance between the threads of the screw in order that the greatest possible mechanical advantage of the combination may be A?

The required distance is equal to

$$\frac{2\pi l}{A}.$$

2. In a weighing machine, the arms of the lever AO (see fig. Art. 111) are 8 feet and 1 foot, the arms of the lever KL are 8 feet and 6 inches, and the arms of the balance are 14 inches and 1 inch; what weight on the platform will be sustained by 1 oz. in the scale-pan?

The weight sustained will be 1 cwt.

3. In a Hunterian screw, the larger screw has 10 threads to an inch, and the smaller screw has 11 threads to an inch, and the length of the lever is 10 inches; required the mechanical advantage.

The mechanical advantage is 6911·5 nearly.

4. In the common screw press, determine the pressure upon the threads when a power P acts at the extremity of the lever.

Let a be the advantage of the press, and b that of the lever, then the required pressure is equal to

$$P\sqrt{(a^2+b^2)}.$$

5. In a crane, the number of teeth in the two wheels are 30 and 40, the number of leaves in the two pinions are 6 and 10, the arm of the winch is 2 feet, and the radius of the weight-bearing axle is 3 inches; required the mechanical advantage.

The advantage is 160.

6. Determine the pressures on the three axes of a crane when the power and the weight are both at right angles with the line joining the axes.

Let a be the advantage of the winch and axle, then the pressure on their axis is equal to

$$P\,(a+1);$$

let b be the advantage of the first wheel and axle, then the pressure on the axis is

$$Pa\,(b+1);$$

and let c be the advantage of the second wheel and axle, then the pressure on their axis is

$$P\,ab\,(c+1).$$

7. If a weight be raised by an endless screw, the mechanical advantage of which is a, show that the space described by the power is equal to a times that described by the weight.

8. In a weighing machine, the advantage of the lever AO (see fig. Art. 111) is a, the advantage of the lever KL is b, and the advantage of the lever MN is c; determine the pressures upon the different fulcra, when a weight W is placed upon the platform so as to press equally upon the four pins E, F, G, H.

The pressures at A, B, C, and D are each equal to

$$\frac{W(a-1)}{4a}.$$

The pressure at L is equal to

$$\frac{W(b-1)}{ab},$$

and the pressure at the fulcrum of MN is equal to

$$\frac{W(c+1)}{abc}.$$

$s = vt$. in uniform velocity.
$v = ft$ in uniform accelerating force.

DYNAMICS.

CHAPTER VII.

ON THE LAWS OF MOTION.

116. When forces not in equilibrium act upon a body, motion must ensue. The degree of rapidity with which the body moves is termed its velocity.

117. Velocity may be either uniform or variable. Velocity is uniform when the body describes *all* equal spaces in equal times, and may therefore be measured by the space described in a unit of time. It is customary to take a second for the unit of time, and a foot for the unit of measure. Thus, if a body moving with a uniform velocity have passed through 1000 feet in 8 seconds, its velocity is $1000 \div 8$, or 125; and generally, if s be the space in feet through which a body has passed in t seconds, then if v denote the velocity,

$$v = \frac{s}{t}, \text{ or } s = vt.$$

Velocity is variable when *all* equal spaces are *not* described in equal times. When a body in motion has a variable velocity, its velocity at any moment may be measured by the space which would be described in a unit of time, if at that moment the velocity were to cease to vary.

118. First Law of Motion. *A body in motion, not acted on by any external force, will continue to move in a straight line, and with a uniform velocity.*

This is equivalent to the assertion, that matter possesses no inherent power of changing the direction or the state of its motion.

The truth of this law must be decided by an appeal to experiment. The powers with which matter has been endowed can evidently be learnt only by observation. Every attempt to prove their existence by *à priori* demonstration will be found to assume in some way or other the property under consideration.

The first law of motion cannot, however, be established by any *direct* experiment, for the prescribed conditions can under no circumstances be fulfilled. No where can we find a body which is not acted on by some external force. Every particle of matter is subject to a variety of external influences. Thus, if a ball be rolled along the ground, it is acted on by the attraction of all surrounding matter, by the resistance of the atmosphere, and by the force of friction; and it moves neither in a straight line, nor with a uniform velocity. It is found, however, that the more we lessen the influence of external force, the more nearly does the motion become direct and uniform.

If a ball be rolled along a smooth pavement, it will move for a longer time and in a line more nearly straight than when thrown with the same velocity along a rough road; and still more so if rolled along a sheet of ice.

If a weight suspended by a thread from any point be made to oscillate, it will after a time come to rest. One of the external forces acting upon the body is the resistance of the air. If this be diminished by causing the body to move within the exhausted receiver of an air pump, the oscillation will continue for a longer period; and the more perfect the vacuum the longer will the motion continue.

On a railway, after a train has acquired the desired velocity, it is no longer necessary for the engine to work, except so far as

to overcome the effects of external forces, such as the friction of the rails and the resistance of the atmosphere.

The most convincing evidence of the truth of this law is found in the accordance of the consequences deducible from it with observed phenomena. It is impossible to doubt the correctness of a principle, upon the assumption of which the motions of the moon can be predicted with almost unerring certainty, and the time of an eclipse foretold within the fraction of a second.

119. It follows from the first law of motion, that if a force continue to act upon a body, the velocity of the resulting motion will undergo continual change.

On this account force, when generating motion in a body, is termed *accelerating force*, and that too whether its effect be to increase or to diminish the velocity of the body.

If the change in the velocity resulting from the action of any force be uniform; that is, if in *all* equal periods of time there be an equal increase or an equal diminution in the velocity, then the force is called a *uniform accelerating force*, and the force itself is measured by the velocity it generates (or destroys) in a unit of time. Thus, when we speak in Dynamics of a force 32, we mean a force which generates in each second of time a velocity of 32 feet per second; that is, in one second it generates a velocity 32, in two seconds a velocity 64, in three seconds a velocity 96, and so on. And hence, generally, if f represent any force measured, as just explained, by the velocity it generates per second, then if v be the velocity at the expiration of t seconds,

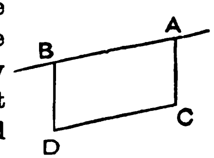
$$v = ft.$$

120. Forces which act during an extremely short time only, as, for example, when a body is set in motion by a sudden blow, or a ball is fired from a cannon, are termed *impulsive forces*. It will be seen hereafter, that although under the action of such a force, it sometimes appears that motion is communicated *instantaneously*, yet this is not really the case. The force, in fact, acts through

a finite though short interval of time, and throughout that period the velocity of the body undergoes continual change. These successive changes, however, take place with such immense rapidity, that they cannot be brought under examination. We concern ourselves therefore only with the velocity and direction of the motion which takes place after the force has ceased.

121. SECOND LAW OF MOTION. *When a force acts upon a body in motion, the change of motion produced is the same, both in magnitude and direction, as if the force acted on the body at rest.*

Thus, if a body move along the line AB with such a velocity that it would describe the space AB in one second; and if, when it arrives at A, a force act upon it such as of itself to cause a body to pass from A to C in one second, then at the end of the second the body will be found at D; the change of motion represented by BD being the same in magnitude and direction as if the force had acted upon the body when at rest. Each force produces its full effect in its own direction.



This law is proved by such experiments as the following:—

If a ball, lying on the deck of a vessel moving steadily through the water at a uniform rate, be struck with a blow of any magnitude and direction, its motion along the deck is the same as if the vessel had been at rest.

If a stone be dropped from the top of the mast, when a vessel is moving uniformly in any direction, and with any velocity, it will fall at the foot of the mast, just as it would if the vessel had been at rest.

If from any point a ball be let fall, and another ball be at the same instant projected forward horizontally with any velocity whatever, both balls will strike the ground at the same time. Here the ball at rest and the ball in motion are acted upon by the same vertical force; namely, the force of gravity, and both are caused to pass through the same vertical space in the same time.

If a person in a railway carriage throw a ball perpendicularly upwards, it will not fall towards the back of the carriage, but will drop into the hands of the individual who projected it.

122. The force acting upon the moving body, in these and similar experiments, is either an impulsive force, or the force of gravity. Within the limits taken in the experiments, the force of gravity does not sensibly vary either in magnitude or direction. It is, therefore, only when a force acts for an indefinitely short time; or, if acting throughout a measurable interval, is uniform both in magnitude and direction, that the second law of motion can be regarded as established by such experiments as those described above. It is important that the student give particular attention to this, inasmuch as the general terms in which it has become customary to enunciate this law might otherwise lead him into error.

Newton's Second a force

123. THE THIRD LAW OF MOTION. *When ~~pressure~~ communicates motion to a body, the accelerating force varies as the ratio of the ~~pressure~~ and mass.*

acceleration

force

Thus, if a pressure of 12 lbs. communicates motion to a mass weighing 20 lbs., and a pressure of 8 lbs. communicates motion to a mass weighing 10 lbs., the third law of motion declares that the force when 12 lbs. moves 20 lbs. : the force when 8 lbs. moves 10 lbs.

$$:: \frac{12}{20} : \frac{8}{10}.$$

124. The force of gravity, usually represented by g, is found by experiment to be a uniform accelerating force, generating a velocity of about 32 feet per second (more accurately 32·2).

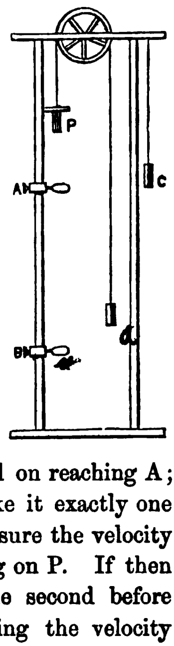
When a body falls by its own weight, the force which produces motion is the force of gravity, and hence by an easy deduction from the third law of motion, as stated above, we obtain the accelerating force when a pressure P communicates motion to a mass

M; for force when P moves M : force when P moves P (that is, g) :: $P \div M : P \div P$. Therefore

$$\text{force when P moves } M = g\frac{P}{M}.$$

It is under this form that the third law of motion is most conveniently verified experimentally. The mode in which this is done is explained in the following article.

125. Attwood's Machine. This consists of two pillars, one of which is graduated, supporting a pulley arranged so as to work with as little friction as possible. An open ring A and a stage B slide along the graduated pillar, and by means of screws can be fixed at any part of it. P and Q are two equal cylindrical weights, connected by a cord passing over the pulley. C is a pendulum beating seconds.



P and Q, being equal, will have of themselves no tendency to motion; but if a small bar be placed upon P, P will descend with an accelerated velocity, until it reach the ring A, which, allowing P to pass through, but intercepting the bar, removes the cause of acceleration. P will then, in accordance with the first law of motion, move uniformly with the velocity it had acquired on reaching A; and if the stage B be so placed that P may strike it exactly one second after reaching A, the distance AB will measure the velocity generated during the interval the bar was resting on P. If then the bar be allowed to rest on P for exactly one second before reaching A, we obtain the means of determining the velocity generated in one second; that is, the measure of the accelerating force. This will always be found to be that given by the formula

$$f = g\frac{P}{M}.$$

Thus, if the two weights are together equal to $15\frac{3}{4}$ oz., and if the bar weigh $\frac{1}{4}$ oz., then, since the pressure communicating motion is $\frac{1}{4}$ oz., and the mass moved is 16 oz., the accelerating force is affirmed by the third law, as expressed in the formula just quoted, to be $g \div 64$, or $\frac{1}{2}$; that is, it generates per second a velocity of $\frac{1}{2}$ ft. or 6 in. If then P, with the bar attached, fall for one second before reaching A (and it will be found by trial that in order to this P must move from a point just 3 in. above A), then the distance AB, over which it moves in the next second, will be found to be 6 in.

126. *Def.* The product of the mass of any body, and the accelerating force acting upon it, is called the *moving force* of the body. moving force $= fM$ acc $\propto \frac{P}{M}$, moving force $\propto P$

Hence the third law of motion may be enunciated thus: *When pressure communicates motion to a body, the moving force varies as the pressure.*

For, let a pressure P communicate motion to a body M, and a pressure P′ to a body M′; then if f and f' be the accelerating forces,

$$f : f' :: \frac{P}{M} : \frac{P'}{M'}.$$

Therefore, $$\frac{fP'}{M'} = \frac{f'P}{M};$$

or, $$fMP' = f'M'P;$$

whence, $$fM : f'M' :: P : P'.$$

---

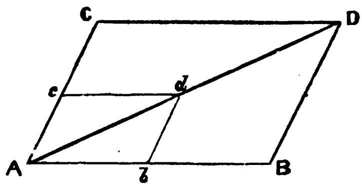
l = length v = velocity
t = time f = acceleration

$v = \frac{l}{t}$, $f = \frac{l}{t^2}$

CHAPTER VIII.

ON THE COMPOSITION AND RESOLUTION OF VELOCITIES.

127. *If* two uniform velocities *be simultaneously impressed upon a body, the resultant motion is* uniform and in a straight line; *and if two sides of a parallelogram represent the impressed velocities in magnitude and direction, the diagonal will represent the magnitude and direction of the resultant velocity.*



Let a body move uniformly along AB, with a velocity which would carry it from A to B in the time T. When the body is at A, let a velocity be impressed upon it such as of itself would cause the body to move uniformly from A to C in the time T; then, drawing the parallelogram ABDC, by the second law of motion, D is the position of the body at the end of the time T.

The motion of the body is along the straight line AD; for let t denote any time whatever, and take Ab so that

$$\mathrm{A}b : \mathrm{AB} :: t : \mathrm{T};$$

then, since the velocity in the direction AB is uniform, Ab is the distance over which the body would move along AB in the time t.

Complete the parallelogram Abdc, then, by similarity of triangles,

$$bd : \mathrm{BD} :: \mathrm{A}b : \mathrm{AB}.$$

Therefore, $\quad \mathrm{A}c : \mathrm{AC} :: \mathrm{A}b : \mathrm{AB},$

or, $\quad \mathrm{A}c : \mathrm{AC} :: t : \mathrm{T}.$

Therefore Ac is the distance over which the body would move along AC in the time t; and hence, by the second law of motion, the body will be found at d, a point in the line AD. And this is true for all values of t; therefore the body will always be found in the line AD.

Again, the motion along AD is uniform; for

$$\mathrm{A}d : \mathrm{AD} :: \mathrm{A}b : \mathrm{AB};$$

or,

$$\mathrm{A}d : \mathrm{AD} :: t : \mathrm{T};$$

or the space described is proportional to the time; that is, the motion is uniform.

Also, if AB, AC represent the spaces over which the impressed velocities would carry the body in one second, then AD is the space actually described by the body in one second; that is, the diagonal represents the magnitude of the resultant velocity.

128. *Conversely;* if any uniform velocity be represented by the line AD, it may be resolved into two velocities, in the direction AB and AC, whose magnitudes are represented by the lines AB and AC. For if any other lengths than AB and AC be taken to represent the component velocities, the resultant velocity will not be represented by AD.

129. It follows, from Art. 127, that if any number of uniform velocities be simultaneously impressed upon a body, the resultant motion will be uniform and in a straight line. The magnitude and direction of the resultant motion will be found by first taking the resultant of any two, then of this resultant and a third, and so on, until all the velocities have been compounded into one.

130. *If* any two velocities, *whether uniform or variable, be simultaneously impressed upon a body, the actual velocity of the body* at that instant *will be represented in magnitude and direction by the diagonal of the parallelogram whose sides represent the impressed velocities.*

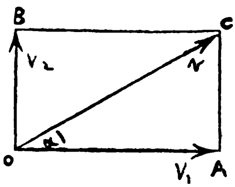
This has been already proved, when both of the impressed velocities are uniform.

If one or both are variable, then let the lines AB and AC (fig. Art. 127) represent the velocities with which the bodies would have moved, if at the instant of reaching A all acceleration of the velocity were to cease. These then being uniform velocities, the line AD will represent, upon the same hypothesis, the actual velocity of the body at the same instant; that is, if all acceleration were to cease, the body would move in the direction AD with a velocity represented in magnitude by the line AD; and when a body moves with a variable velocity, the velocity at any instant is measured, both in magnitude and direction, by the path the body would describe in the succeeding second, if at that instant all variation, both in magnitude and direction, were to cease.

131. Hence, if the velocity of a moving body at any instant be represented by any straight line in magnitude and direction, it may be resolved into two other velocities, represented in magnitude and direction by the sides of any parallelogram which has the given line for its diagonal.

132. The most frequent application of the preceding propositions is in the case in which the component velocities act in directions at right angles to each other.

Thus, let two velocities V_1 and V_2, simultaneously impressed upon a body at O, in directions at right angles to each other, be represented by the lines OA and OB; then, if v be the resultant velocity, v is represented by the line OC. But since $OC^2 = OA^2 + OB^2$,



$$v^2 = V_1^2 + V_2^2;$$

and if the angle COA be α,

$$\tan \alpha = \frac{V_2}{V_1}.$$

Again, if OC represent any velocity v, and it be required to resolve this velocity into two velocities, acting in directions at right angles to each other, namely, in the directions OA and OB; then if V_1 and V_2 be these components,

$$V_1 = v \cos a,$$

and $$V_2 = v \sin a.$$

133. *If two* uniformly accelerated velocities *be simultaneously impressed upon a body, the resultant motion will be* in a straight line, and with a uniformly accelerated velocity; *and if the sides of a parallelogram represent the magnitudes and directions of the impressed velocities, the diagonal will represent the magnitude and direction of the resultant velocity.*

Let two velocities be impressed upon a body at A (fig. Art. 127), one of which would, if acting alone, cause the body to move from A to B in T seconds, with a velocity whose acceleration is f_1 feet per second; and the other of which would, if acting alone, cause the body to move from A to C in T seconds, with a velocity whose acceleration is f_2 feet per second; then, by the second law of motion, the place of the body at the end of T seconds is at D.

But since the spaces described in equal times by bodies moving with uniformly accelerated velocities, that is, under the action of uniformly accelerating forces, are in proportion to the forces

$$AB : AC :: f_1 : f_2.$$

Let Ab be the space which would be described along AB in any time t, and complete the parallelogram Abdc; then,

$$Ab : Ac :: AB : AC,$$

whence, $$Ab : Ac :: f_1 : f_2;$$

and therefore Ac is the space which would be described in the direction AC in the time t. Consequently, by the second law of motion, the place of the body at the end of any time t is at d; that is, is in the line AD. The motion of the body is therefore in the straight line AD.

The resultant velocity is also uniformly accelerated; for since

$$Ad : Ab :: AD : AB.$$

Ad, the space described by the body in any time t, varies in the same way as Ab, the space described in the same time by a body moving with a uniformly accelerated velocity.

Also, if AB and AC represent the impressed velocities f_1 and f_2, AD shall represent the resultant velocity. Let f be the resultant force; then, since AD is the space described under the action of f, in the same time as the space AB would be described under the action of f_1; therefore,

$$AD : AB :: f : f_1,$$

or AD represents f as AB represents f_1.

134. Since uniform accelerating forces are measured by the velocity generated in one second of time, it follows from the preceding, that two uniform accelerating forces, acting together upon a body, may be compounded into a single uniform accelerating force, represented in magnitude and direction by the diagonal of the parallelogram whose sides represent the magnitude and direction of the component forces. And, conversely, a uniform accelerating force, represented by the diagonal of a parallelogram, may be resolved into two uniform accelerating forces, represented by the sides of the parallelogram.

135. *Def.* If a force act upon a body, and from any cause the motion of the body be not in the direction of the force, then the force which at any instant would, if it acted in the direction of the motion, produce the actual change which takes place, is termed the *effective accelerating force* at that instant; and the force which really acts upon a body is distinguished as the *impressed* accelerating force.

136. *To find the effective accelerating force when a uniform accelerating force* f *acts upon a body constrained to move along a given line.*

Let OC (fig. Art. 132) be the direction of f, and also represent it in magnitude, and let the body be constrained to move along OA; then, completing the parallelogram OACB, the force f may be resolved into forces represented by OA and OB. By the conditions of the problem, all motion in the direction OB is prevented, the latter component may therefore be disregarded. The other component OA acts in the direction of the motion, and is, therefore, the force required. Hence,

$$\text{the effective accel. force} : f :: \text{OA} : \text{OC};$$

or,

$$\text{effective accel. force} = f \cdot \frac{\text{OA}}{\text{OC}}.$$

If a be the angle between the direction of the force and the line of motion, then

$$\text{effective accel. force} = f \cdot \cos a.$$

EXAMPLES.

1. A body moves in a direction inclined to the vertical line, at an angle of 60°, with a velocity of 100 feet; required the horizontal and vertical velocities.

The horizontal velocity is $50\sqrt{3}$ feet, and the vertical velocity is 50 feet.

2. A moving body has, at a given instant, a vertical velocity of 28 feet, and a horizontal velocity of 96 feet; required its actual velocity at that instant.

The required velocity is 100 feet.

3. A moving body has, at a given instant, a vertical velocity of 86·6 feet, and a horizontal velocity of 50 feet; required the direction of its motion at that instant.

The body is moving in a direction inclined to the vertical line at an angle of 30° nearly.

4. Three equal velocities are simultaneously impressed upon a body, one vertically, another horizontally, and the third in a direction inclined to the vertical line at an angle of $30°$; required the actual velocity of the body.

Let each of the impressed velocities be equal to V, then the actual velocity is equal to

$$V \,.\, \sqrt{(4+\sqrt{3})}.$$

5. Upon a body placed at A, one of the corners of a regular hexagon ABCDEF, velocities are simultaneously impressed such as would separately carry the body at a uniform rate to the other corners in three seconds; in what time will the body arrive at D?

The body will arrive at D in one second.

6. In the preceding, if each of the impressed velocities had been equal, what would have been the magnitude of the resultant velocity?

Let V be the magnitude of each of the impressed velocities, then the resultant velocity will be equal to

$$V \,.\, (2+\sqrt{3}).$$

---

117 Paragraph 137. 119 139 147 147

$(s = vt)$ $(s = \frac{1}{2} f t^2)$. $(v = ft)$. $(v^2 = 2fs)$ $(v = V + ft)$ $(s = Vt - \frac{1}{2} ft^2)$

$\left(t = \sqrt{\frac{2h}{g}}\right)$. $(v^2 = 2gh)$.

Paragraph 152 153

CHAPTER IX.

ON UNIFORM ACCELERATING FORCES AND GRAVITY.

137. *To find the space described in* t *seconds, when a body moves from rest under the action of a uniform accelerating force.*

Let f denote the force, and s the space described. Let the time be divided into n equal periods, each of which will consequently be equal to $\frac{t}{n}$. By Art. 119, the velocity at the end of the first of these periods will be $\frac{ft}{n}$, at the end of the second $\frac{2ft}{n}$, at the end of the third, $\frac{3ft}{n}$, and so on.

If during these intervals the body be supposed to move uniformly with the velocity acquired at the end of each, then (since space = vel. × time, when velocity is uniform) the space described in the first interval will be $\frac{ft^2}{n^2}$, in the second $\frac{2ft^2}{n^2}$, and so on; the sum of this series will be the whole space described. Therefore,

$$s = \frac{ft^2}{n^2} + \frac{2ft^2}{n^2} + \frac{3ft^2}{n^2} + \text{\&c.} \ldots\ldots + \frac{nft^2}{n^2}$$

$$= \frac{ft^2}{n^2}(1 + 2 + 3 + \text{\&c.} \ldots\ldots + n)$$

$$= \frac{ft^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{ft^2}{2} + \frac{ft^2}{2n}.$$

The error arising from supposing the velocity to be uniform

through each of these intervals will diminish, if the length of the intervals be diminished; that is, if n increase; and the more n increases, the more nearly shall we approach to the actual space described. As n increases, the value of $\frac{ft^2}{2n}$ diminishes without limit; and, therefore,

$$s = \tfrac{1}{2}ft^2.$$

138. By the preceding, the space described in n seconds is $\frac{1}{2}fn^2$, and the space in $n-1$ seconds is $\frac{1}{2}f(n-1)^2$; the difference between these is the space described in the nth second; therefore,

$$\begin{aligned}\text{space in } n\text{th second} &= \tfrac{1}{2}f\{n^2-(n-1)^2\},\\ &= \tfrac{1}{2}f(2n-1).\end{aligned}$$

But $2n-1$ is the nth odd number, therefore the space described in the nth second equals $\frac{1}{2}f$ multiplied by the nth odd number.

139. *To find the relation between the space described and the velocity acquired by a body moving from rest under the action of a uniform accelerating force.*

By Art. 119, $v=ft$, and by Art. 137, $s=\frac{1}{2}ft^2$. Squaring both sides of the former, and multiplying both sides of the latter by $2f$, we obtain

$$v^2 = f^2t^2,$$

and

$$2fs = f^2t^2,$$

$\therefore$

$$v^2 = 2fs.$$

140. The three equations obtained in Articles 119, 137, and 139; namely,

$$\begin{aligned}v &= ft,\\ s &= \tfrac{1}{2}ft^2,\\ v^2 &= 2fs,\end{aligned}$$

express the relation between each pair of the three quantities; viz., the space described, the velocity acquired, and the time when a body moves from rest under the action of a uniform accelerating

force. If any one of these quantities be known, either of the other two may, by means of one or other of these equations, be found directly. When gravity is the force considered, these equations are written

$$v = gt,$$
$$s = \tfrac{1}{2}gt^2,$$
$$v^2 = 2gs.$$

141. The following are examples of the application of these formulæ to cases in which a body falls freely under the action of gravity.

Ex. 1. A stone falls from rest under the action of gravity; find the space described in 5 seconds.

In the formula, $s = \frac{1}{2}gt^2$, make $t = 5$; then

$$s = 16 \cdot 1 \times 25 \quad = 402 \cdot 5 \text{ feet.}$$

Ex. 2. Find the velocity which a stone will acquire by falling through 1610 feet.

In the formula, $v^2 = 2gs$, make $s = 1610$; then

$$v^2 = 64 \cdot 4 \times 1610,$$
$$= 103684;$$
$$\therefore \quad v = \sqrt{103684} = 322.$$

Ex. 3. How long must a body fall under the action of gravity to acquire a velocity of 128·8 feet per second?

In the formula, $v = gt$, make $v = 128 \cdot 8$; then

$$128 \cdot 8 = 32 \cdot 2 \times t;$$
$$\therefore \quad t = 128 \cdot 8 \div 32 \cdot 2 = 4 \text{ seconds.}$$

Ex. 4. How far must a body fall under the action of gravity to acquire a velocity of 96·6?

In the formula, $v^2 = 2gs$, make $v = 96 \cdot 6$; then

$$(96 \cdot 6)^2 = 64 \cdot 4 \times s;$$
$$\therefore \quad s = 144 \cdot 9.$$

142. *If the space described under the action of a uniform accelerating force be divided into any number of parts, the square of the time of describing from rest the whole space is equal to the sum of the squares of the times of describing from rest the several parts.*

Let the space described in the time T, under the action of any force f, be $a_1 + a_2 + \&c. \dots + a_n$; then, since $s = \frac{1}{2}ft^2$,

$$T^2 = \frac{2}{f}(a_1 + a_2 + \&c. \dots + a_n).$$

But if t_1, t_2, &c., be the times of describing from rest the distance a_1, a_2, &c., then

$$t_1^2 = \frac{2a_1}{f},$$

$$t_2^2 = \frac{2a_2}{f},$$

$$t_n^2 = \frac{2a_n}{f}.$$

Substituting these in the first equation, we have

$$T^2 = t_1^2 + t_2^2 + \&c. \dots + t_n^2.$$

COR. Hence, if T be the time of describing the whole distance, and t_1 the time of describing from rest any part of the distance, then, if t_2 be the time of describing from rest the remaining part,

$$t_2^2 = T^2 - t_1^2.$$

Ex. 1. A stone can fall from a certain height to the ground in 10 seconds; if it be arrested at the end of 8 seconds, and afterwards let fall, how long will it be in descending the remaining distance?

Let t be the time required, then, by the proposition just demonstrated,

$$t^2 = 10^2 - 8^2,$$
$$= 36,$$
or, $$t = 6.$$

Ex. 2. A stone falls to the ground in 6 seconds; how long was it in passing over the third tenth of the entire distance?

Let t be the time of describing the first tenth, then by the proposition given above,

$$10\,t^2 = 36,$$

or,
$$t = \frac{3\sqrt{10}}{5}.$$

Hence, the time of describing three-tenths of the distance is

$$\frac{3\sqrt{10}\sqrt{3}}{5},$$

and the time of describing two-tenths is

$$\frac{3\sqrt{10}\sqrt{2}}{5};$$

therefore the time of describing the third tenth is

$$\tfrac{3}{5}\,(\sqrt{30}-\sqrt{20}).$$

143. *To find the tension in the string and the accelerating force when a weight* P, *hanging freely, draws a weight* Q *along a smooth horizontal plane.*

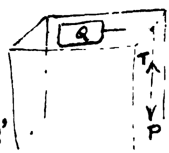
Let T be the tension in the string, and f the accelerating force.

The pressure causing motion in P is P − T; therefore, by Art. 124,

$$f = g\,.\,\frac{P-T}{P}.$$

The pressure causing motion in Q is T; therefore, also,

$$f = g\,.\,\frac{T}{Q}.$$



force causing motion in all = P

$\therefore f = g\,\frac{P}{P+Q}$

force causing motion in Q = T

$\therefore f = g\,\frac{T}{Q}$

$\therefore T = \frac{PQ}{P+Q}$

Solving these equations, we obtain

$$T = \frac{PQ}{P+Q},$$

and
$$f = g\,.\,\frac{P}{P+Q}.$$

144. *To find the accelerating force and the tension in the string when a weight* P *draws a weight* Q *over a fixed pulley, neglecting the inertia of the pulley.*

Let T be the tension in the string, and f the accelerating force.

The pressure causing motion in P is $P-T$; therefore, by Art. 124,

$$f=g\,\frac{P-T}{P}.$$

The pressure causing motion in Q is $T-Q$; therefore

$$f=g\,\frac{T-Q}{Q}.$$

Solving these equations, we obtain

$$T=\frac{2PQ}{P+Q}.$$

and hence,

$$f=g\cdot\frac{P-Q}{P+Q}.$$

force causing motion in all $= P-Q$

$\therefore\ f = g\cdot\frac{P-Q}{P+Q}.$

force causing motion in Q $= T-Q$

$\therefore\ f = g\ \frac{T-Q}{Q}$

$\therefore\ T = \frac{2PQ}{P+Q}.$

145. The following are examples in illustration of the preceding.

Ex. 1. A weight of 9 ounces draws a weight of 7 ounces over a fixed pulley; find the space described from rest in t seconds, neglecting the inertia of the pulley.

By Art. 144, the accelerating force $=\dfrac{g(9-7)}{9+7}=4$ nearly.

Substituting this in the general formula $s=\frac{1}{2}ft^2$, we have

$$s=2t^2.$$

Hence, in one second the space described is 2 feet, in two seconds 8 feet, in three seconds 18 feet, and so on.

Ex. 2. A weight of 9 lbs. is drawn along a smooth horizontal table by a weight of 1 lb. hanging vertically by a string passing over a pulley at the edge of the table; find the space described from rest in 3 seconds, the velocity acquired in 4 seconds, and the space described in the 5th second.

By Art. 143, the accelerating force $=\dfrac{g\times 1}{9+1}=3\cdot 2$;

therefore, the space described in 3 seconds $=1\cdot 6\times 9=14\cdot 4$ feet, and the velocity acquired in 4 seconds $=3\cdot 2\times 4=12\cdot 8$.

The space described in the 5th second is $\frac{1}{2}f$ multiplied by the 5th odd number, or 9; therefore,

$$\text{the space described in 5th second} = 1{\cdot}6 \times 9 = 14{\cdot}4 \text{ feet.}$$

Ex. 3. A weight of 10 lbs. draws a weight of 8 lbs. over a fixed pulley; required the tension in the string, and the time of raising the lighter weight to a height of 64 feet.

By Art. 144,

$$\text{the tension} = \frac{2 \times 10 \times 8}{10 + 8} = 8\tfrac{8}{9} \text{ lbs.}$$

Also, if f be the accelerating force,

$$f = g \cdot \frac{10 - 8}{10 + 8} = \frac{g}{9}.$$

Substituting this value for f, and 64 for s, in the equation $s = \frac{1}{2}ft^2$, we have

$$64 = \frac{g}{18}t^2;$$

$$\therefore \quad t^2 = 36 \text{ nearly,}$$

$$\text{or,} \quad t = 6 \text{ seconds.}$$

146. *If a body be projected with a given velocity* V, *and be acted on in the same direction by a uniform accelerating force* f, *to find the velocity acquired and the space described in a given time* t.

From the second law of motion, it follows that the velocity will equal the velocity of projection together with the velocity generated by the force; and, therefore, if v denote the velocity required,

$$v = V + ft.$$

And similarly, the space described is equal to the space due to the velocity of projection together with the space due to the action of the force. The space described by a uniform velocity V in t seconds is equal to Vt, and the space due to the action of a uniform accelerating force in t seconds is equal to $\frac{1}{2}ft^2$. Therefore

$$s = Vt + \tfrac{1}{2}ft^2.$$

$\therefore v^2 = V^2 + 2Vft + f^2t^2 = V^2 + 2f(Vt + \frac{1}{2}ft^2)$

$= V^2 + 2fs$

147. If a body be projected with a given velocity V in a direction opposite to that in which the accelerating force acts, the expressions deduced, as in the preceding Article, will be

$$v = V - ft,$$
$$s = Vt - \tfrac{1}{2}ft^2.$$

Squaring both sides of the first equation, we have

$$\begin{aligned} v^2 &= V^2 - 2Vft + f^2t^2, \\ &= V^2 - 2f(Vt - \tfrac{1}{2}ft^2), \\ &= V^2 - 2fs. \end{aligned}$$

148. *To find the height to which a body will rise when projected vertically with a given velocity* V.

Here the direction of V is opposite to that of the force of gravity, or g, and hence, if v denote the velocity acquired when the body has described the space s, by the preceding Article,

$$v^2 = V^2 - 2gs.$$

But when the body has attained the highest point, its velocity at that instant will be zero; and, therefore, if s be the height of this point,

$$o = V^2 - 2gs;$$
$$\therefore \quad 2gs = V^2. \quad s = \frac{V^2}{2g}$$

By comparing this result with the expression $v^2 = 2gs$ in Art. 139, we see that the height to which the body will rise is the same as the distance through which it must fall to acquire the velocity of projection.

149. *To find the time during which a body will rise when projected vertically with a given velocity* V.

Since at the instant of attaining the greatest height $v = o$, then (Art. 147),

$$V - gt = o;$$
$$\therefore \quad gt = V,$$
$$\therefore \quad t = \frac{V}{g}.$$

From the general formula $v = gt$, it follows that $\frac{V}{g}$ is also the time during which a body must fall to acquire the velocity V, and hence from this and the preceding section we see that, on its return to the point whence it was projected, the body will have a velocity equal to the velocity of projection, and that the times of descent and ascent will be equal. *

150. The following are examples in illustration of Articles 146–9.

Ex. 1. A body is projected vertically upwards with a velocity of 100 feet per second, to find how high it will rise.

By Art. 148, $2gs = V^2$; therefore in this case,

$$s = \frac{10000}{64 \cdot 4} = 155 \cdot 28.$$

Ex. 2. A body is projected vertically upwards with a velocity of 161 feet per second, to find how long it will continue to ascend.

By Art. 149, $t = \frac{V}{g}$, therefore in this case,

$$t = \frac{161}{32 \cdot 2} = 5 \text{ seconds.}$$

Ex. 3. A stone is projected vertically upwards, and returns to the same spot after an interval of 12 seconds; find the velocity of projection, and the height to which the body has risen.

The velocity of projection is equal to the velocity acquired by the body during the time of descent, or 6 seconds; and, therefore, if V be the velocity of projection, since $v = gt$,

$$V = 32 \cdot 2 \times 6 = 193 \cdot 2 \text{ feet per second.}$$

The height to which the body has risen is equal to the distance through which it falls from rest during six seconds; that is,

$$\text{the height required} = \tfrac{1}{2} g \times 6^2,$$
$$= 16 \cdot 1 \times 36 = 579 \cdot 6 \text{ feet.}$$

* Art. 148 reduces to this: "To find the distance through which a given body must fall to acquire a given velocity V."
Art. 149 reduces to this "To find the time during which a body must fall to acquire a given velocity V."

Ex. 4. A ball is projected vertically with a velocity 160 feet per second, and two seconds afterwards another ball is projected in the same direction with a velocity 224; when and at what height will the balls meet?

Let the balls meet x seconds after the projection of the second ball, then $x+2$ is the time the first ball has been moving.

The space described by the first ball in $x+2$ seconds is (Art. 147,)

$$160(x+2)-16(x+2)^2.$$

Similarly, the space described by the second ball in x seconds is

$$224x-16x^2.$$

But since the balls meet, these two quantities must be equal;

$$\therefore \quad 224x-16x^2=160(x+2)-16(x+2)^2;$$

$$\therefore \quad 128x=256;$$

$$\therefore \quad x=2;$$

or the balls meet 2 seconds after the projection of the second ball. To find the height, substitute 2 for x in either of the expressions for the space described by the balls; *e.g.*, $224x-16x^2$, then

$$\text{height}=224\times 2-16\times 4=384.$$

151. *To find the force which accelerates a body down an inclined plane.*

Let a weight W rest upon an inclined plane, whose length is l and whose height is h. W would be kept at rest on this plane by a force acting along the plane equal to $W\frac{h}{l}$ (Art. 100). Therefore $W\frac{h}{l}$ is the pressure which causes the body to move down the plane. Hence, by the third law of motion,

$$\text{accelerating force}=g\,.\,\frac{W\frac{h}{l}}{W}=\frac{gh}{l}.$$

152. *Required the time of descent when a body falls down an inclined plane.*

By the general formula, $s = \frac{1}{2}ft^2$. Let h = height of the plane, and l = its length. Then, in this case, $s = l$, and by the preceding Article, $f = \frac{gh}{l}$, and therefore

$$l = \frac{gh}{2l} \cdot t^2;$$

$$\therefore \quad t^2 = \frac{2l^2}{gh},$$

$$\therefore \quad t = l\sqrt{\left(\frac{2}{gh}\right)}.$$

153. *To find the velocity acquired by a body falling down an incline plane.*

By the general formula, $v = \sqrt{(2fs)}$. As in the preceding Article, $s = l$, and $f = \frac{gh}{l}$. Therefore

$$v = \sqrt{(2gh)}.$$

But $\sqrt{(2gh)}$ expresses the velocity which a body acquires in falling freely under the action of gravity through a distance h; therefore the velocity acquired by a body falling down an inclined plane is the same as that acquired in falling through the perpendicular height. Consequently, whatever be the length of the plane, the velocity acquired is the same, if the height remains the same.

154. The results of the three preceding Articles may be also deduced as follows :—

Let α be the inclination of the plane, then the angle between the direction of gravity and plane is $90° - \alpha$; and therefore, by Art. 137, the effective accelerating force is

$$f = g \sin \alpha.$$

To find the time of descent, substitute this value for f, and l for s, in the general formula $s = \frac{1}{2} f t^2$; then

$$l = \tfrac{1}{2} g \sin \alpha \, . \, t^2;$$

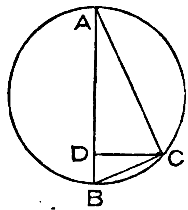
therefore,
$$t = \sqrt{\left(\frac{2l}{g \sin \alpha}\right)},$$
$$= l \sqrt{\left(\frac{2}{gh}\right)}, \quad \text{since } \sin \alpha = \frac{h}{l}.$$

Also, if v be the velocity acquired in falling down the plane, substituting for s and f the values mentioned above in the formula $v = \sqrt{(2 f s)}$,

$$v = \sqrt{(2g \sin \alpha \, . \, l)},$$
$$= \sqrt{(2gh)}.$$

155. *If a circle be placed with its plane vertical, the times of descent down all chords drawn through its highest or lowest points are equal.*

Let ABC be any circle whose plane is vertical. Let AB be the vertical diameter, and AC any chord drawn through A. Let t be the time in which a body falls down AC, then (Art. 152),



$$t = \sqrt{\left(\frac{2 \, AC^2}{g \, . \, AD}\right)}.$$

The triangles ACD, ACB are similar, therefore

$$AD : AC :: AC : AB, \text{ or } \frac{AC^2}{AD} = AB;$$

$$\therefore \qquad t = \sqrt{\left(\frac{2 \, AB}{g}\right)}.$$ = const whatever be length or θ

As this result is independent of the position of the point C, it follows that the times of descent down all chords drawn through A are equal.

And similarly, if BC be any chord drawn through B, the time of descent is

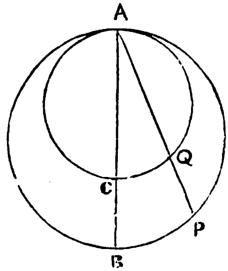
$$\sqrt{\left(\frac{2 \, . \, BC^2}{g \, . \, BD}\right)}.$$

But $\frac{BC^2}{BD} = AB$; therefore, as before, the time of descent is

$$\sqrt{\left(\frac{2 \,.\, AB}{g}\right)}. = \text{const.}^t \text{ wherever } C \text{ may be on } O.$$

156. *If two circles, the planes of which are vertical, touch one another internally at their highest points, and any chord of the larger circle be drawn through the point of contact, the time of descent down that part of the chord which is exterior to the smaller circle is the same for all such lines.*

Let the two circles touch one another internally at A, and let A be their highest point. Let AB be a diameter of the larger circle, and AP any chord.



Let T be the time of descent down AP, and t the time of descent down AQ, then, by Art. 142, the time of descent down QP is equal to

$$\sqrt{(T^2 - t^2)}.$$

But, by Article 155, T is the time of descent down AB, and t is the time of descent down AC; therefore the time of descent down CB is equal to

$$\sqrt{(T^2 - t^2)}.$$

Hence, the time of descent down QP is equal to the time of descent down CB.

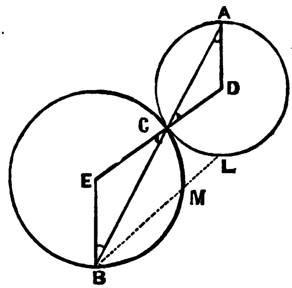
157. In a similar manner it may be shown, that if two circles touch one another internally in their *lowest* points, and any chord of the larger circle be drawn through the point of contact, the time of descent down that portion of the chord which is exterior to the smaller circle is invariable.

158. By aid of the property of the circle demonstrated in

Art. 155, many problems relating to planes of quickest and slowest descent may be readily solved. The following are examples.

PROB. 1. *To find the line of quickest descent from a circle to a point without it, the point and the circle being in the same vertical plane, and the point lower than the highest point of the circle.*

Let AC be the given circle, having its centre at D, and its highest point at A, and let B be the given point. Join the points A, B; the line BC is the line required.



Through B draw a line parallel to AD, and meeting DC produced in E. Then, since the triangle ACD is isosceles,

$$\angle \text{ ACD} = \angle \text{ CAD};$$

and, since the lines BE, AD are parallel,

$$\angle \text{ EBC} = \angle \text{ CAD};$$

therefore, $$\angle \text{ EBC} = \angle \text{ ACD} = \angle \text{ ECB};$$

and $$\text{EC} = \text{EB}.$$

Hence, a circle described with E as a centre and EB as a radius will touch the given circle in C; and, since EB is vertical, it will have its lowest point at B.

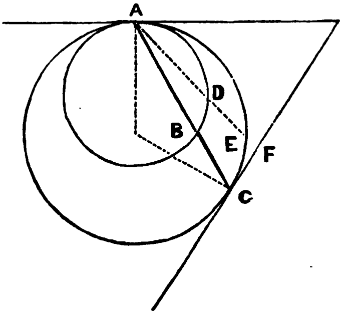
Consequently, the time of descent down CB is less than that down any other line drawn from B to the given circle. For let LB be some other line. The time of descent down CB is equal to the time of descent down MB, and is therefore less than the time of descent down LB.

PROB. 2. *To find the line of quickest descent from a given circle to a given line without it, the line and the circle being in the same vertical plane.*

Let ABD be the given circle, having its highest point at A, and CF the given line.

Through A draw AC, making equal angles with CF and the horizontal line through A; then BC shall be the line required.

Through A and C draw a circle ACE, having its highest point at A. It can easily be shown that this circle will touch the line CF in C.



By the previous problem, BC is the line of quickest descent from the given circle to the point C. In like manner DF is the line of quickest descent from the given circle to any other point F in the line CF.

The time of descent down DE is less than the time of descent down DF; but, by Article 156, the time of descent down BC is equal to that down DE. Therefore, the time of descent down BC is less than down DF. Consequently, BC is the line of quickest descent from the given circle ABD to the given line CF.

159. *To find the tension in the string, and the accelerating force, when a weight* P, *hanging freely, draws a weight* W *up an inclined plane (the string connecting them passing along the plane); and also the time in which* P *will draw* W *along the whole length of the plane.*

Let T be the tension in the string, and f the accelerating force.

The pressure causing motion in P is P − T; hence, by Article 124,

$$f = g \cdot \frac{P-T}{P} = g\left(1 - \frac{T}{P}\right).$$

The pressure of W in the direction of the plane is $W\frac{h}{l}$ (Art. 100), and the pressure causing motion in W is $T - W\frac{h}{l}$; hence,

$$f = g\frac{T - W\frac{h}{l}}{W} = g\left(\frac{T}{W} - \frac{h}{l}\right).$$

force causing motion in all = $P - W\cdot\frac{h}{l}$

$\therefore f = g\dfrac{P - W\frac{h}{l}}{P+W} = g\dfrac{Pl - Wh}{l(P+W)}$

Equating these two values of f, we have

$$\frac{T}{W} - \frac{h}{l} = 1 - \frac{T}{P};$$

force causing motion in P = $P - T$

$\therefore f = g\left(1 - \frac{T}{P}\right)$

or,

$$\frac{T(P+W)}{PW} = \frac{h+l}{l},$$

$\therefore \dfrac{Pl - Wh}{l(P+W)} = 1 - \dfrac{T}{P}$

and $\therefore$

$$T = \frac{PW(h+l)}{(P+W)\,l}.$$

$\therefore T = \dfrac{PW(h+l)}{(P+W)\,l}$.

Substituting this value for T in the first equation, we obtain

$$f = \frac{g\,(Pl - Wh)}{l\,(P+W)}.$$

Let t = the time in which P will draw W along the whole length of the plane. In the general formula, $s = \frac{1}{2}ft^2$; for s substitute l, the length of the plane, and for f the value just determined; then

$$l = g\cdot\frac{Pl - Wh}{l(P+W)}\cdot\frac{t^2}{2},$$

$$t = l\sqrt{\left(\frac{2}{g}\cdot\frac{P+W}{Pl - Wh}\right)}.$$

EXAMPLES.

1. If a body be projected vertically upwards with a velocity of 100 feet per second, when will it attain the height of 156 feet? ✓

In 3 and $3\frac{1}{4}$ seconds.

2. A body projected vertically upwards is at the same height at the expiration of 4 and 5 seconds, what was its greatest elevation? ✓

The greatest elevation is 326 feet.

3. A body projected vertically upwards is at the same height at the expiration of 2 and 3 seconds, required the velocity of projection.

The velocity of projection is 80·5 feet.

4. A ball is projected vertically upwards with a velocity of 160 feet per second, when must another ball be projected in the same direction with a velocity 224, in order that the balls may meet at the height of 384 feet?

The second ball must be projected 2 or 4 seconds *after* the first, or 6 or 8 seconds *before*.

5. Two balls are projected vertically at the same instant, one upwards with a velocity V_1, and the other down with a velocity V_2, when will they be equally distant from the point of projection?

The time required is $\dfrac{V_1 - V_2}{g}$ seconds.

6. A ball is projected horizontally with a velocity V, required its velocity at the end of t seconds.

The required velocity is $\sqrt{(V^2 + g^2t^2)}$.

7. A body slides by its own weight down a perfectly smooth inclined plane, rising 5 in 48; what is the space described in 3 seconds?

The space described is 15 feet.

8. A body slides by its own weight down a smooth inclined plane, whose length is 100 feet and height 9; what is the time of its descent?

The time of descent is $8\frac{1}{3}$ seconds.

9. A body is projected with a velocity of 16·1 feet per second, up a smooth inclined plane rising 1 in 5, required the time and length of its ascent.

The time of ascent is $2\frac{1}{2}$ seconds, and the length $20\frac{1}{8}$ feet.

10. A body is projected up a smooth inclined plane, whose length is 30 feet and height 8 feet; what must be the velocity of projection along the plane that it may just reach the top of the plane?

The required velocity is 22·7 nearly.

11. Two weights, W_1 and W_2, placed upon a double inclined plane, are connected by a string passing over the summit; what is the accelerating force, when W_1 draws up W_2, the lengths of the planes being l_1 and l_2, and the height h?

The accelerating force is equal to $\frac{gh}{l_1 l_2} \cdot \frac{W_1 l_2 - W_2 l_1}{W_1 + W_2}$.

12. In the preceding, let $W_1 = 30$, $W_2 = 10$, $l_1 = 60$, $l_2 = 75$, $h = 48$; what is the time in which W_1 moves down its own plane?

The time required is 2·92 seconds.

13. From a given point draw a straight line down which a body will descend to a fixed line in the shortest time possible.

14. Two circles, the planes of which are vertical, are so placed that the lowest point of one is in contact with the highest point of the other; show that the time of descent from any point in the former to any point in the latter, along a straight line passing through the point of contact, is invariable.

v = resultant vel

$(v^2 = V^2 + g^2t^2 - 2Vgt \sin\alpha)$. $\left(\sin\theta = \frac{V\cos\alpha}{v}\right)$ $(\text{Elevation} = Vt\sin\alpha - \frac{1}{2}gt^2)$.
$\left(\text{time of flight} = 2 \cdot \frac{V\sin\alpha}{g}\right)$ $\left(\text{horiz. range} = Vt\cos\alpha = \frac{V^2\sin 2\alpha}{g}\right)$

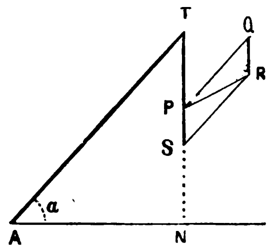
CHAPTER X.

ON PROJECTILES.

160. DEF. The *angle of elevation* of a projectile is the inclination of the direction of projection to the horizontal plane through the point of projection.

161. *A body is projected with a given velocity* V, *at a given angle of elevation* α, *to determine the magnitude and direction of its velocity at the end of* t''.

Let A be the point, and AT the direction of projection. Make $AT = Vt$, and through T draw TP vertically downwards, and equal to $\frac{1}{2}gt^2$, or the space due to the action of gravity in t''. Then, by the second law of motion, P will be the place of the body at the end of t''.



Through P draw PQ parallel to AT, and equal to V, and PS, vertically downwards, and equal to gt, or the velocity generated by gravity in t''. Complete the parallelogram QS, then, by Art. 130, the diagonal PR represents the required velocity in magnitude and direction. Let v represent this velocity, and θ the angle between its direction and the downward vertical line, then

$$v^2 = PQ^2 + QR^2 - 2PQ \,.\, QR \cos PQR,$$
$$= V^2 + g^2t^2 - 2Vgt \sin\alpha.$$

Time of attaining least vel. of a projectile. v is least when $v = V\cos\alpha$
thus $V^2+g^2t^2-2Vgt\sin\alpha = V^2\cos^2\alpha$
$\therefore g^2t^2-2Vgt\sin\alpha+V^2\sin^2\alpha=0$
$\therefore gt - V\sin\alpha = 0$
$\therefore t = \frac{V\sin\alpha}{g}$

Also,

$$\sin\theta = \frac{PQ \sin PQR}{PR},$$

$$= \frac{V\cos\alpha}{v}. \quad v = \frac{V\cos\alpha}{\sin\theta}$$

$\therefore$ v is least when $\sin\theta$ is greatest, i.e. $=90°$ or equals $V\cos\alpha$

162. *To determine the least velocity of a projectile, and the time of attaining it.*

Let V be the velocity of projection, and α the angle of elevation, then, by the preceding, the square of the velocity at any time t is

$$V^2 + g^2t^2 - 2Vgt \sin\alpha.$$

By substituting $V^2(\cos^2\alpha + \sin^2\alpha)$ for V^2, this expression may be written in the form

$$V^2\cos^2\alpha + (gt - V\sin\alpha)^2.$$

This expression has its least value when its second term is equal to zero, or $gt = V\sin\alpha$; hence,

$$\text{least velocity} = V\cos\alpha, \quad \text{in vert. proj.}^n \text{ least } v = 0$$

$$\text{time of reaching it} = \frac{V\sin\alpha}{g}. \quad \text{, in vert. proj.}^n\ t = \frac{V}{g} \ \S 149.$$

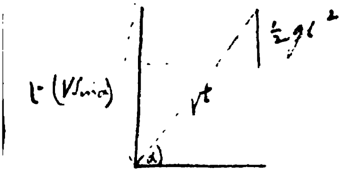
Cor. The value of the least velocity just found is equal to the horizontal component of the velocity of projection; whence it may be inferred that the body is moving in a horizontal direction at the instant of attaining its least velocity. This may also be deduced from the expression already found for determining the direction of the velocity. For, substituting V cos α for v in the expression

$$\sin\theta = \frac{V\cos\alpha}{v}, \text{ we have}$$

$$\sin\theta = \frac{V\cos\alpha}{V\cos\alpha} = 1,$$

$$\therefore \qquad \theta = 90°;$$

or the direction of the velocity is inclined to the vertical line at an angle of 90°; that is, is horizontal.



163. *To determine the elevation of a projectile at any given instant of time.*

Let V be the velocity of projection, and a the angle of elevation. Then in fig. Art. 161, PN represents the elevation at the end of t''.

$$\text{Elevation} = TN - TP,$$
$$= Vt \sin a - \tfrac{1}{2} gt^2.$$

In vert. proj. elevation $= Vt - \frac{1}{2}gt^2$ cf. § 147.

164. *To determine the greatest elevation of a projectile and the time of reaching it.*

By the preceding,

$$\text{elevation} = \frac{2Vgt \sin a - g^2t^2}{2g}.$$
$$= \frac{V^2 \sin^2 a - (V \sin a - gt)^2}{2g}.$$

This expression has its greatest value when the second term is equal to zero or $gt = V \sin a$; hence

$$\text{the greatest elevation} = \frac{V^2 \sin^2 a}{2g}, \quad = \frac{V^2}{2g} \text{ in vert. proj}^n \text{ Comp. § 148}$$

$$\text{the time of reaching it} = \frac{V \sin a}{g}. \quad = \frac{V}{g} \quad \text{Comp. § 149}$$

Cor. By a comparison of this result with Art. 162, it is seen that at the instant of reaching the greatest elevation, the projectile is moving with its least velocity.

165. Defs. The *range* of a projectile is the distance between the point of projection and the point where the body strikes any plane passing through the point of projection; and the *time of flight* is the time the body takes in describing its path between these two points.

166. *To find the time of flight and the range of a projectile on a horizontal plane.*

Let V be the velocity of projection, and a the angle of elevation. Then, by Art. 163, the elevation of the body at the time t is

$$Vt \sin a - \tfrac{1}{2} gt^2.$$

This expression is equal to zero, or the body is in the horizontal plane through the point of projection when either $t = 0$, or $t = \dfrac{2V \sin a}{g}$. The former of these values denotes the instant of projection, the latter, the instant of returning to the horizontal plane. Hence,

$$\text{time of flight} = \frac{2V \sin a}{g}.$$

From fig. Art. 161, it will be seen that AN, or the horizontal distance described in the time t, is equal to $Vt \cos a$; therefore,

$$\begin{aligned} \text{horizontal range} &= V \cos a \times \text{time of flight}, \\ &= \frac{2V^2 \sin a \cos a}{g}, \\ &= \frac{V^2 \sin 2a}{g}. \end{aligned}$$

COR. 1. By comparing the results of the present section with those of Art. 164, it will be seen that the time of flight is double the time of reaching the greatest elevation, and consequently that the times of ascent and descent are equal. Also, that the ratio of the horizontal range to the greatest elevation is constant for the same angle of elevation, being equal to

$$4 \cot a.$$

COR. 2. Since $\sin 2a$ has its greatest value when $a = 45°$, it follows that, with a given velocity of projection, the horizontal range is the greatest when the angle of elevation is $45°$, and hence, V being the velocity of projection,

$$\text{greatest horizontal range} = \frac{V^2}{g}.$$

167. Some of the preceding results may be simplified by the following device: Let h represent the distance through which a

body must fall under the action of gravity to acquire the velocity of projection, then, Art. 139, $V^2 = 2gh$, and hence

$$\begin{aligned} \text{greatest elevation} &= h \sin^2\alpha, \\ \text{horizontal range} &= 2h \sin 2\alpha, \\ \text{greatest horizontal range} &= 2h. \end{aligned}$$

168. *Given the time of flight of a projectile, on a horizontal plane, to find the greatest elevation.*

Let T be the time of flight, then

$$\frac{2V \sin \alpha}{g} = T,$$

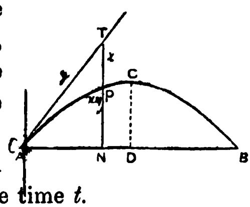
$$V \sin \alpha = \tfrac{1}{2} gT;$$

$$\text{greatest elevation} = \frac{V^2 \sin^2\alpha}{2g}, \quad = \frac{gT^2}{8}.$$

Hence all projectiles, having equal times of flight on a horizontal plane, attain to equal elevations; and, conversely, if the greatest elevations are equal, the times of flight are equal also.

169. *To determine the path described under the action of gravity by a body projected with a given velocity in a given direction.*

Let the body be projected from A in the direction AT, with the velocity V. Let AT be the distance due to the velocity of projection in any time t, whence $AT = Vt$. Draw the vertical line through T, and let $TP = \frac{1}{2}gt^2 =$ the space due to the action of gravity in the time t. Then, by the second law of motion, P will be the place of the body at the end of the time t.



Let y and x be the co-ordinates of P, referred to AT and the vertical line through A as axes. Then

$$y = AT = Vt,$$
$$x = TP = \tfrac{1}{2}gt^2;$$

$\therefore$ eliminating t $$y^2 = \frac{2V^2}{g}x,$$

or the path of the projectile is a parabola.

170. *To find the equation to the path of a projectile when the axes of co-ordinates are the horizontal and vertical lines drawn through the point of projection.*

Let AN, NP (fig. Art. 169) be the co-ordinates of P, and let $AN = y$, $NP = x$, and the angle $TAN = \alpha$,

$$y = AT \cos\alpha = Vt \cos\alpha,$$
$$x = TN - TP = Vt \sin\alpha - \tfrac{1}{2}gt^2;$$

$\therefore$ eliminating t,

$$x = \frac{\sin\alpha \, . \, y}{\cos\alpha} - \frac{gy^2}{2V^2\cos^2\alpha},$$

or
$$y^2 - \frac{V^2 \sin 2\alpha}{g} \, . \, y = -\frac{2V^2\cos^2\alpha}{g} \, . \, x;$$

whence it appears that the path of the projectile is the parabola whose *latus rectum* is

$$\frac{2V^2\cos^2\alpha}{g}.$$

171. If, as before, h be the distance through which a body must fall vertically under the action of gravity to obtain the velocity V, then since $V^2 = 2gh$, the equation to the path of the projectile becomes

$$y^2 - 2h \sin 2\alpha \, . \, y = -4h \cos^2\alpha \, . \, x.$$

Again, if h' be the distance through which a body must fall to acquire a velocity equal to the horizontal component of the velocity of projection, or $V \cos\alpha$, so that $V^2\cos^2\alpha = 2gh'$, the preceding equation becomes

$$y^2 - 4h' \tan\alpha \, . \, y = -4h' \, . \, x.$$

Hence, the *latus rectum* of the parabola described by a projectile is four times the space through which a body must fall to acquire the horizontal component of the velocity of projection.

172. The values already obtained (Art. 167) for the greatest elevation and the horizontal range, may also be deduced imme-

diately from the equation to the path of the projection. By Art. 171,

$$y^2 - 2h \sin 2\alpha \,.\, y = -4h \cos^2 \alpha \,.\, x;$$

$$\therefore \quad (y - h \sin 2\alpha)^2 = -4h \cos^2 \alpha \,.\, x + h^2 \sin^2 2\alpha,$$

$$= -4h \cos^2 \alpha \,(x - h \sin^2 \alpha),$$

which is the equation to a parabola, having its vertex at a point whose co-ordinates are $h \sin^2 \alpha$ and $h \sin 2\alpha$. Therefore (fig. Art. 169),

$$CD = h \sin^2 \alpha,$$

$$AD = h \sin 2\alpha;$$

but CD is the greatest elevation, and AD is half the horizontal range.

173. *The distance of the point of projection from the directrix of the parabolic path is equal to the space due to the velocity of projection.*

The distance of the point of projection from the directrix is equal to the distance between the directrix and the horizontal plane through the point of projection. The *latus rectum* of the parabolic path is $4h \cos^2 \alpha$, and consequently the distance of the directrix from the vertex is $h \cos^2 \alpha$. But the distance of the vertex from the horizontal plane is $h \sin^2 \alpha$; therefore

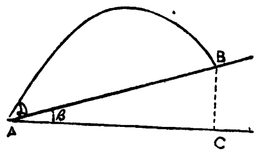
$$\text{height of directrix} = h \cos^2 \alpha + h \sin^2 \alpha,$$

$$= h.$$

Any point in the path of a projectile may be regarded as the point of projection, if the body be supposed to be projected from it with the velocity and in the direction which it has at that point. Hence it follows from the preceding, that the velocity of a projectile at any point in its path is equal to that acquired by a body falling to the point from the directrix.

174. *To find the range and time of flight of a projectile on an inclined plane.*

Let the body be projected from A with a velocity V, in a direction making an angle α with the horizontal line. Let β be the inclination of the plane. By the equation to the path of the parabola (Art. 171),



$$AC^2 - 2h \sin 2\alpha . AC = -4h \cos^2 \alpha . BC.$$

But AB $\cos \beta =$ AC, and AB $\sin \beta =$ BC. Therefore, substituting

$$AB \cos^2\beta - 2h \sin 2\alpha \cos \beta = -4h \cos^2 \alpha \sin \beta;$$

$$\therefore \quad AB = \frac{2h \sin 2\alpha \cos \beta - 4h \cos^2 \alpha \sin \beta}{\cos^2 \beta},$$

$$= \frac{4h \cos \alpha . \sin (\alpha - \beta)}{\cos^2 \beta},$$

which is the range required.

Let T be the time of flight; then, as in Art. 170,

$$AC = V \cos \alpha . T.$$

But AC = AB $\cos \beta$, therefore

$$T = \frac{4h . \sin (\alpha - \beta)}{V \cos \beta},$$

$$= \frac{2V \sin (\alpha - \beta)}{g \cos \beta}.$$

175. *To find the direction in which a body must be projected with a given velocity in order to strike a given fixed mark.*

Let B (fig. Art. 174) be the mark, and let AC $= b$, and BC $= c$. Let V be the velocity of projection, and α the angle its direction makes with the horizontal line. Then, if the mark be struck at the end of the time t,

$$V \cos \alpha . t = b,$$

and $$V \sin \alpha . t - \tfrac{1}{2} g t^2 = c.$$

Therefore, eliminating t,

$$b \tan \alpha - \tfrac{1}{2} \frac{g b^2}{V^2 \cos^2 \alpha} = c.$$

Whence, $$b \tan a - \frac{g}{2}\frac{b^2}{V^2}(1 + \tan^2 a) = c,$$

$$gb \tan^2 a - 2V^2 . \tan a + \frac{2V^2c + gb^2}{b} = 0;$$

$$\therefore \qquad \tan a = \frac{V^2 \pm \sqrt{(V^4 - 2V^2cg - g^2b^2)}}{gb}.$$

Or, if $2gh = V^2$,

$$\tan a = \frac{2h \pm \sqrt{(4h^2 - 4hc - b^2)}}{b}.$$

EXAMPLES.

1. The horizontal range of a projectile is 1000 feet, and the time of flight 10 seconds; required the velocity of projection.

The required velocity is 189·6 feet per second.

2. The greatest elevation attained by a projectile is equal to its horizontal range; required the angle of elevation.

The required angle is $\tan^{-1}4$.

3. In what direction must a body be projected with a velocity V, that its range on a plane whose inclination is β may be the greatest possible?

The required angle of elevation is equal to $45° + \frac{1}{2}\beta$.

4. Show that the greatest possible range, on a plane whose inclination is β, of a body projected with a velocity due to the height h is

$$\frac{2h}{1 + \sin \beta}.$$

5. The horizontal range of a projectile is equal to the space due to the velocity of projection; what is the angle of elevation?

The angle of elevation is $15°$.

K 2

6. If a body be projected from a height h' above a horizontal plane, with a velocity due to the height h, and at an angle of elevation equal to α, what is its range along the plane?

The range is equal to

$$h \sin 2\alpha + 2 \cos \alpha \sqrt{(h^2 \sin^2 \alpha + hh')}.$$

7. If any number of bodies be projected with the same velocity v, in different directions from the same point, show that their locus at the end of the time t is the sphere whose radius is vt, and whose centre is at a distance $\frac{1}{2}gt^2$ vertically below the point of projection.

8. Two bodies projected from the same point, at angles of elevation α_1, α_2, respectively, and with velocities due to the heights h_1, h_2, strike an inclined plane at the same point, find the inclination of the plane.

The inclination of the plane is equal to

$$\tan^{-1}\left(\frac{1}{2} \cdot \frac{h_1 \sin 2\alpha_1 - h_2 \sin 2\alpha_2}{h_1 \cos^2 \alpha_1 - h_2 \cos^2 \alpha}\right).$$

9. Two bodies projected from the same point, at angles of elevation α_1, α_2, respectively, and with velocities due to the heights h_1, h_2, strike the same point in a vertical wall, find the distance of the wall from the point of projection.

The required distance is equal to

$$4h_1h_2 \cdot \frac{\tan \alpha_1 - \tan \alpha_2}{h_2 \sec^2 \alpha_1 - h_1 \sec^2 \alpha_2}.$$

10. A ball is projected with a velocity of 966 feet; what must be the angle of elevation that its horizontal range may be 9660 yards?

The required angle is 45°.

11. The greatest horizontal range of a projectile is 805 feet; what was its velocity of projection?

The required velocity is 161 feet per second.

12. Given the horizontal range and the time of flight to determine the angle of elevation.

Let R be the horizontal range and T the time of flight, then, if α be the angle of elevation,

$$\tan \alpha = \frac{gT^2}{2R}.$$

13. If any number of bodies be projected from the same point in the same direction, but with different velocities, show that the locus of their points of greatest elevation is a straight line through the point of projection inclined to the horizontal line at an angle whose tangent is $\frac{1}{2}$ tan α (α being the angle of elevation of the projectiles).

14. Two bodies are projected at the same instant from different points in the same horizontal plane; the sines of the angles of elevation are inversely proportional to the velocities of projection; show that their elevations are equal at every instant of their flight.

15. If two bodies, projected at the same instant, from different points in the same horizontal plane, have at any instant equal elevations, show that their elevations are equal at every instant.

16. If any number of bodies be projected from the same point with the same velocity in different directions in the same vertical plane, show that the locus of their points of greatest elevation is an ellipse whose semi-major axis is h and semi-minor axis $\frac{1}{2}h$ (h being the height through which a body must fall to acquire the velocity of projection).

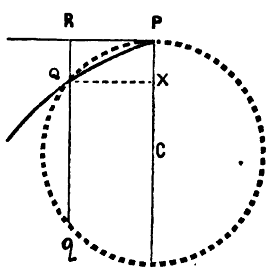
17. In the preceding, show that the locus of the foci of the parabolic paths is a circle whose centre is at the point of projection, and whose radius is equal to h.

CHAPTER XI.

ON THE FREE CURVILINEAR MOTION OF A PARTICLE, AND ON MOTION ABOUT CENTRES OF FORCE.

176. By the first law of motion, a body in motion, unacted upon by any force, will move on in a straight line. Curvilinear motion can therefore result only from the continued action of one or more forces, and these forces are either partly or wholly employed in overcoming at each instant the tendency of the body to rectilinear motion. Let a single force be conceived of as producing, *in this respect only*, the same effect as the force or forces really acting upon the body. The direction of this force must at each instant be at right angles with the tangent to the path in which the body is moving. A force equal and opposite to this is taken to represent the tendency of the body to rectilinear motion, and is called the *centrifugal force.*

177. *To find the centrifugal force when a body is moving in a curvilinear orbit with a velocity* v.



Let the body when moving with the velocity v be at the point P, and let PR be the tangent and PC the normal at P. Let f be the force which acts upon P in the direction of the normal, then, by definition, the centrifugal force is a force equal and opposite to f. Let Q be position of the body after a short interval t, and draw QR parallel to PC, then if f be supposed to be uniform during the interval t,

the second law of motion is applicable (see Art. 122), and PR is the space due to the velocity at P, and PX or QR the space due to the force f. Therefore upon this supposition,

$$PR = vt,$$

and $$QR = \tfrac{1}{2} ft^2,$$

whence $$f = \frac{2\,QR\,v^2}{PR^2} \text{ = in limit } \frac{2v^2}{2\rho} = \frac{v^2}{\rho}$$

The error arising from the supposed uniformity of f is diminished the nearer Q is taken to P; hence, ultimately,

$$f = \text{limit of} . \frac{2\,QR\,v^2}{PR^2}.$$

To determine the value of this limit, let the circle PQq be drawn passing through the points P, Q, and touching the line PR in P. Let RQ produced meet the circle in q. Then, *Euclid*, iii. 36,

$$PR^2 = RQ \,.\, Rq,$$

$$\therefore \quad \frac{RQ}{PR^2} = \frac{1}{Rq}.$$

But, as Q approaches P, Rq becomes more and more nearly equal to the diameter of the circle, and the circle itself to the circle of curvature. (See Appendix.) Hence, if ρ be the radius of curvature,

$$\text{limit of } \frac{RQ}{PR^2} = \text{limit of } \frac{1}{Rq},$$

$$= \frac{1}{2\rho}.$$

Therefore,

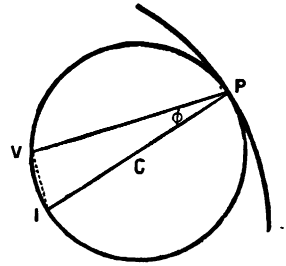
$$f = \frac{v^2}{\rho}.$$

COR. 1. Hence, if F be the force which acts upon a particle moving freely in a curvilinear orbit, and ϕ be the angle between the direction of F and the normal at any point P, then

$$F \cos \phi = \frac{v^2}{\rho}.$$

or, $$F = \frac{v^2}{\rho \cos \phi}.$$

COR. 2. Let PV be the chord of curvature, (*i. e.*, the chord of the circle of curvature,) drawn through P in the direction of the force, then $PV = PI \cos\phi = 2\rho\cos\phi$, and hence by the preceding corollary,



$$v^2 = F \cdot \frac{PV}{2},$$

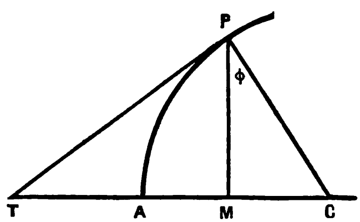
$$= 2F \cdot \frac{PV}{4}.$$

Comparing this with the formula, $v^2 = 2fs$, Art. 139, we see that when a body moves in a curvilinear orbit under the action of any force, the velocity at any point is that which would be acquired by a body falling through one-fourth of the chord of curvature drawn in the direction of the force, under the action of the force at that point supposed to be uniform.

178. *To determine the law of force when a particle moves in a circular orbit under the action of a force at right angles to one of the diameters.*

Let AM be the given diameter, P any position of the particle, and F the force at P. Then, if G be the centre of the circle, and r its radius,

$$F = \frac{v^2}{r\cos\phi}.$$



But, by hypothesis, the force always acts in a direction at right angles to AM, it can therefore produce no acceleration in the direction of AM, and hence the velocity of the particle in this direction must be constant. Let V represent this constant velocity, and through P draw the tangent PT, then PT is the direction of

the velocity v, and $v \cos \text{PTM}$ is its component in the direction of AM, therefore,

$$V = v \cos \text{PTM},$$
$$= v \cos \phi;$$

whence,

$$F = \frac{V^2}{r \cos^3\phi},$$
$$= \frac{V^2 r^2}{r^3 \cos^3\phi},$$
$$= \frac{V^2 r^2}{\text{PM}^3},$$

$$\therefore \quad F \propto \frac{1}{\text{PM}^3},$$

or the force varies inversely as the cube of the distance from the diameter AG.

Cor. Since $v = \dfrac{V}{\cos \phi}$ and $\cos \phi = \dfrac{\text{PM}}{r}$, we have

$$v = \frac{Vr}{\text{PM}},$$

$$\therefore \quad v \propto \frac{1}{\text{PM}};$$

or the velocity varies inversely as the distance of the particle from the diameter AG.

179. *To determine the law of force when a particle moves in a parabolic orbit under the action of a force at right angles to the axis.*

In the fig. of preceding Art. let AM be the axis of the parabola, PG the normal, and PT the tangent at P, then as before,

$$F = \frac{v^2}{\rho \cos \phi},$$

and

$$v = \frac{V}{\cos \phi},$$

whence

$$F = \frac{V^2}{\rho \cos^3 \phi}.$$

Let $4c$ be the *latus rectum* of the parabola, and let r be the focal distance of P; then (see Appendix)

$$\rho = 2c\left(\frac{r}{c}\right)^{\frac{3}{2}}$$

and $$\cos\phi = \frac{\mathrm{PM}}{2\sqrt{(cr)}},$$

$\therefore$ $$\rho\cos^3\phi = \frac{\mathrm{PM}^3}{4c^2},$$

and $$\mathrm{F} = \frac{4\mathrm{V}^2c^2}{\mathrm{PM}^3};$$

or $$\mathrm{F} \propto \frac{1}{\mathrm{PM}^3}.$$

COR. Sin $v = \dfrac{\mathrm{V}}{\cos\phi}$, and $\cos\phi = \dfrac{\mathrm{PM}}{2\sqrt{(cr)}}$,

$\therefore$ $$v = \frac{2\mathrm{V}\sqrt{(cr)}}{\mathrm{PM}}.$$

180. *To determine the law of force when a particle moves in an elliptic orbit under the action of a force at right angles to the major axis.*

Using the same notation as in the preceding, we have as before

$$\mathrm{F} = \frac{\mathrm{V}^2}{\rho\cos^3\phi}.$$

Let a and b be the semi-axes of the ellipse, and r, r' the focal distances of P, then (see Appendix)

$$\rho = \frac{(rr')^{\frac{3}{2}}}{ab},$$

and $$\cos\phi = \frac{a}{b}\cdot\frac{\mathrm{PM}}{\sqrt{(rr')}}.$$

Therefore, $$\mathrm{F} = \frac{\mathrm{V}^2\,b^4}{a^2\,\mathrm{PM}^3},$$

or $$\mathrm{F} \propto \frac{1}{\mathrm{PM}^3}.$$

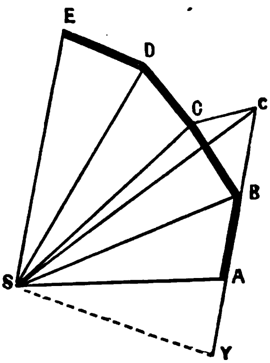
COR. Since $v = \frac{V}{\cos\phi}$, and $\cos\phi = \frac{a}{b} \cdot \frac{PM}{\sqrt{(rr')}}$;

$$v = \frac{Vb\sqrt{(rr')}}{aPM}.$$

181. A *central force* is a force which always tends towards a fixed point or centre. Central forces, when attractive, are sometimes called *centripetal* forces.

182. "*If a body move in any orbit about a fixed centre of force, the areas described by lines drawn from the centre to the body lie in one plane, and are proportional to the times of describing them.*" (Newton, *Principa;* book i. sect. 2, prop. i.)

Let S be the centre of force; and suppose a body unattracted by the force in S to describe the straight line AB with a uniform velocity in τ''. Then, if suffered to proceed, it would move on uniformly in the direction of AB produced, and describe $Bc = AB$ in the next interval of τ''; but at B suppose an instantaneous impulse communicated to it in the direction BS, which causes it to move in the direction BC; then, by the second law of motion, if Cc be drawn parallel to BS, the body at the end of the second interval will be found at C. Draw SA, SB, SC, Sc. Since Cc is parallel to BS, the triangles SBC, SBc, are equal; and since $Bc = AB$, the triangles SBc, SAB are equal. Therefore the triangles SBC, SAB are equal, and are in the same plane, since no force has acted to draw the body out of the plane SAB. Similarly, if impulses be communicated at the end of every interval of τ'', in directions tending always to S, causing the body to describe CD, DE, &c. in the third, fourth, &c. intervals, the triangles SCD, SDE, &c. will



be each equal to SAB, and in the same plane with it; also their bases AB, BC, CD, &c. are described in equal times. Therefore, the area of any number of these triangles, or the polygon SABCDE varies as the time of describing it. Now let the number of intervals be increased, and the magnitude of each diminished indefinitely, then the polygon approximates to a curvilinear area, the sum of the impulses to a continued force always tending to S as their limits, and what was proved of those quantities is true of their limits, and therefore the curvilinear area described in any time is proportional to the time.

COR. 1. If v be the velocity of the body at A, and p the perpendicular from S upon the tangent at A, then

$$\text{area described in } t'' = \tfrac{1}{2}ptv.$$

For, draw SY perpendicular to AB, then since AB is ultimately the tangent, SY is ultimately equal to p. Let $t = n\tau$, and, as before, let AB be the space described in the interval τ with the velocity v, then $AB = v\tau$, and

$$\begin{aligned}\text{area described in } \tau'' &= \tfrac{1}{2}\,SY \times AB,\\ &= \tfrac{1}{2}\,SY \times v\tau.\end{aligned}$$

But, by the preceding proposition, the area described in t'' is n times the area described in τ''; therefore,

$$\begin{aligned}\text{area described in } t'' &= \tfrac{1}{2}\,SY \times vn\tau,\\ &= \tfrac{1}{2}\,SY \times vt,\\ &= \tfrac{1}{2}\,pvt \text{ ultimately.}\end{aligned}$$

COR. 2. If h be equal to twice the area described in 1″, then, by preceding corollary,

$$h = pv,$$

and hence

$$v = \frac{h}{p},$$

or the velocity at any point varies inversely as the perpendicular upon the tangent at that point.

183. *"If a body moving in a plane curve describe areas proportional to the times by lines drawn from the body to any point, the body is acted on by forces all tending to that point."* (Newton, *Principia;* book i. sect. 2, prop. ii.)

Let S (fig. Art. 182) be the point about which areas proportional to the times are described, and suppose, as in the preceding proposition, that a body, unacted upon by any force, describes the straight line AB in τ''. In AB produced, take $Bc = AB$, then, if suffered to proceed, the body would be at c at the end of the next interval of τ''; but at B suppose an impulse communicated which causes it to describe BC in the second interval, such that the triangle SBC is equal to and in the same plane with the triangle SAB. Join C, c and S, c. Then, since the triangles SBc, SBC are each equal to the triangle SAB, they are equal to each other, and therefore Cc is parallel to SB, and therefore, by the second law of motion, the impulse communicated at B is in the direction BS. Similarly, if D, E, &c., be the places of the body at the ends of the third, fourth, &c., intervals of τ'', so that the triangles SAB, SBC, SCD, &c., are all equal, all the impulses communicated may be shown in like manner to tend to S.

Now, suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the polygon is the curvilinear area, and that of the sum of the impulses a continued force tending to S; therefore the body is acted upon by a force tending to S.

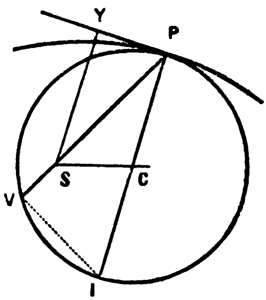
184. *If a body move in any orbit about a centre of force* (S), *and if* ρ *be the radius of curvature at any point* (P), p *the perpendicular from the centre upon the tangent at the given point, and* h *twice the area described in* $1''$, *then if* F *be the central force,*

$$F = \frac{h^2\,SP}{p^3\rho};$$

for, let PG be the normal, and PY the tangent at P, let SY be perpendicular to PY, and the angle $SPG = \phi$. Then, by Art. 177, Cor. 1,

$$F = \frac{v^2}{\rho\cos\phi}.$$

But (Art. 182, Cor. 2) $v = \frac{h}{p}$; therefore,



$$F = \frac{h^2}{p^2 \rho \cos \phi}$$

$$= \frac{h^2 \, SP}{p^2 \rho \, SP \cos \phi},$$

$$= \frac{h^2 \, SP}{p^3 \rho}, \text{ since } SP \cos \phi = p.$$

By aid of this expression, the law of force can be readily found, when the orbit in which a body is moving about a centre of force is given. The following sections are examples of its application to such problems.

185. *To find the law of force when a body moves in a circle under the action of a force tending to a given point.*

Let S (fig. Art. 184) be the given point, and PVI the given circle whose radius PG $= a$. Then, in this case, $\rho = a$, and therefore

$$F = \frac{h^2 \, SP}{p^3 \, a}.$$

Let PS, or PS produced, meet the circle in V; then, since SP $\cos \phi = p$, and $2a \cos \phi = PV$,

$$p = \frac{SP \, . \, PV}{2a},$$

whence

$$F = \frac{8h^2 a^2}{SP^2 \, . \, PV^3};$$

or,

$$F \propto \frac{1}{SP^2 \, . \, PV^3}.$$

186. *To find the law of force when a body moves in an ellipse under the action of a force tending to the centre.*

If a, b are the semi-axes of the ellipse, and r, r' the focal distances of any point, then (see Appendix)

$$\rho = \frac{(rr')^{\frac{3}{2}}}{ab}$$

and since S is at the centre,

$$p = \frac{ab}{\sqrt{(rr')}},$$

$$\therefore \qquad p^3\rho = a^3b^3,$$

$$\therefore \qquad F = \frac{h^2 . SP}{a^2b^2};$$

or, $$F \propto SP,$$

that is, the force varies directly as the distance.

Cor. 1. Hence if μ be the value of F at the unit of distance, or, as it is termed, the absolute force, then

$$F = \mu . SP,$$

and $$\mu = \frac{h^2}{a^2b^2}.$$

Cor. 2. Let P be the periodic time, or the time of describing the complete ellipse, then, by Art. 182,

$$\frac{P}{1''} = \frac{\text{area of ellipse}}{\text{space described in } 1''},$$

$$P = \frac{2\pi ab}{h};$$

but if μ be the absolute force, $h = ab\sqrt{\mu}$, therefore,

$$P = \frac{2\pi}{\sqrt{\mu}},$$

or the periodic time is independent of the dimensions of the ellipse, and hence, if several bodies move in different ellipses about the same force in the centre, the times of their revolutions will be the same.

187. *To find the law of force when a body moves in an ellipse under the action of a force tending towards one of the foci.*

Let a and b be the semi-axes of the ellipse, r and r' the focal distances of any point, the former being measured from that focus towards which the force is directed; then (see Appendix)

$$p = b\sqrt{\left(\frac{r}{r'}\right)},$$

and as before,

$$\rho = \frac{(rr')^{\frac{3}{2}}}{ab},$$

$$\therefore \qquad p^3\rho = \frac{b^2r^3}{a} = \frac{b^2.\mathrm{SP}^3}{a},$$

$$\therefore \qquad \mathrm{F} = \frac{ah^2}{b^2}\cdot\frac{1}{\mathrm{SP}^2},$$

$$\text{or,} \qquad \mathrm{F} \propto \frac{1}{\mathrm{SP}^2}$$

Cor. If P be the periodic time, then, as before,

$$\mathrm{P} = \frac{2\pi ab}{h};$$

but if μ be the absolute force, $\mu = \frac{ah^2}{b^2}$; and, therefore,

$$\mathrm{P} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

Hence, if several bodies move in different ellipses, about the same force in the focus, the squares of the periodic times are as the cubes of the major axes.

188. The values of p and ρ being the same for the hyperbola as for the ellipse, it follows that the investigations of Arts. 186, 187, will serve also for the cases of a body moving in a hyperbolic orbit about a force directed severally to the centre and the focus.

189. *To find the law of force when a body moves in a parabolic orbit about a force tending towards the focus.*

Let $4c$ be the *latus rectum* of the parabola, and r the focal distance of any point, then (Appendix)

$$\rho = 2c\left(\frac{r}{c}\right)^3,$$

and $$p = \surd(cr),$$

$\therefore$ $$p^3\rho = 2cr^3 = 2c\,.\,SP^3.$$

$\therefore$ $$F = \frac{h^2}{2c}\cdot\frac{1}{SP^2};$$

or $$F \propto \frac{1}{SP^2}.$$

COR. Since $v = \dfrac{h}{p}$, and $p = \surd(cr)$, therefore,

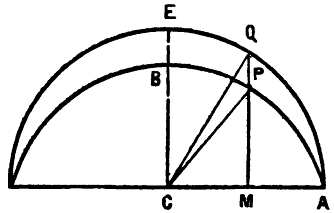
$$v = \frac{h}{\surd(cr)},$$

or, $$v^2 \propto \frac{1}{r},$$

that is, the square of the velocity varies inversely as the distance.

The following problem is of especial importance in the theory of undulations.

190. *A body moves in an elliptic orbit under the action of a force directed towards the centre to find the position of the body at any time* t.



Let APB be the ellipse, P the position of the body at the time t, and, for greater simplicity, suppose the time to be reckoned from the instant when the body was at B. Then (Art. 182),

$$\frac{t}{1''} = \frac{2\text{ area BCP}}{h}.$$

With C as a centre, and CA as a radius, describe the circle

AQE, and let the ordinate through P meet this circle in Q. Let a and b be the semi-axes of the ellipse, and x the abscissa of the point P. Then,

$$\text{area BCMP} = \frac{b}{a}\,\text{area ECMQ},$$

$$= \frac{b}{a}\,(\text{CMQ} + \text{ECQ}),$$

$$= \frac{b}{a}\left(\frac{x\,.\,\text{MQ}}{2} + \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2}\right),$$

$$= \frac{x\,.\,\text{MP}}{2} + \frac{ab \sin^{-1}\left(\frac{x}{a}\right)}{2},$$

$$= \text{area CMP} + \frac{ab \sin^{-1}\left(\frac{x}{a}\right)}{2}.$$

But
$$\text{area BCP} = \text{area BCMP} - \text{area CMP},$$

$\therefore$
$$= \frac{ab \sin^{-1}\left(\frac{x}{a}\right)}{2}.$$

Therefore,

$$t = \frac{ab \sin^{-1}\left(\frac{x}{a}\right)}{h}.$$

But if μ be the absolute force, then (Art. 186, Cor. 1) $h = ab\sqrt{\mu}$, therefore

$$t = \frac{\sin^{-1}\left(\frac{x}{a}\right)}{\sqrt{\mu}},$$

and hence

$$x = a \sin \sqrt{\mu}\,.\,t.$$

COR. As this result is independent of the minor axis of the ellipse, it remains true, whatever value we may give to b. We may, therefore, suppose b to vanish, and we have the case of a body moving in a straight line, under the action of a force varying directly as the distance.

EXAMPLES.

1. To find the law of force when a particle moves in a parabolic orbit under the action of a force parallel to the axis.

The force is a constant force.

2. To find the law of force when a particle moves in a hyperbolic orbit under the action of a force parallel to the major axis.

3. To find the law of force when a particle moves in a parabolic orbit under the action of a force parallel to a given straight line.

Draw a tangent to the parabola parallel to the given line, and let y' be the ordinate of the particle referred to this tangent, and the diameter through the point of contact as axes, then

$$F \propto \frac{1}{y'^3}.$$

4. Several bodies move in parabolic orbits under the action of the same force in the focus, show that the square of the time of moving from the vertex to the extremity of the *latus rectum* varies as the cube of the *latus rectum.*

5. A body moves in a parabola under the action of a force directed to a given point in the axis produced, find the law of force.

Let S be the centre of force, A the vertex of the parabola, P any position of the body, and M the point where the ordinate through P cuts the axis, then

$$F \propto \frac{SP}{(2SA - SM)^3}.$$

6. A body moves in a parabola about a force in the focus, show that the centre of the circle which passes through the body, the

focus, and the vertex, moves uniformly with a velocity equal to 3-8ths of the velocity of the body at the vertex. (Newton, *Princip.* i. prop. xxx.)

7. A body moves in an ellipse under the action of a force directed towards one of the extremities of the major axis, find the law of force.

Let S, the centre of force, be the origin, and SM the abscissa of the point P, then

$$F \propto \frac{SP}{SM^3}.$$

8. A body moves in a circle under the action of a force directed to a point (S) in the circumference, A is the other extremity of the diameter through S, find the time of moving from A to S.

Let μ be the absolute force, and a the radius of the circle, then the time required is

$$\frac{2\pi a^3\sqrt{2}}{\sqrt{\mu}}.$$

9. Two particles move in circular orbits about forces in their centres, the absolute force and the magnitude of the orbits is the same in both cases, but in the one case the force varies as the inverse square of the distance, and in the other as the inverse cube of the distance, compare their periodic times.

Let a be the radius of the orbits, T_1 the periodic time in the first case, and T_2 in the second, then

$$T_2^2 = aT_1^2.$$

HYDROSTATICS.

CHAPTER XII.

ON THE FUNDAMENTAL PROPERTIES OF FLUIDS.

191. A fluid is a body all of whose parts can move freely amongst themselves. Motion consequently can be caused amongst the particles of a fluid body, by the application of the slightest conceivable force.

192. The science which treats of the equilibrium of forces acting upon fluid bodies is termed Hydrostatics.

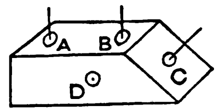
193. Fluids are divided into liquids and aëriform fluids. An aëriform fluid is distinguished from a liquid by the existence of an expansive or repulsive force amongst its particles, in consequence of which they tend in a greater or less degree to move off from one another.

194. When a fluid is at rest, any portion of it may be supposed to become a solid, without disturbing the equilibrium.

Since all the particles of the fluid are at rest, it is clear that they will not be less so if any number of them be rigidly connected with each other, and so deprived of the power of relative motion.

195. *When pressure is communicated to a fluid mass in equilibrium, it is transmitted equally and in all directions.*

In the sides of a closed vessel of any shape, let a number of apertures of equal area be made, and let these apertures be supplied with pistons exactly fitting them. Let the vessel be filled with water, and let the pistons be maintained by some mechanical contrivance in their respective positions.* If any one of the pistons be pressed in with a force P, it is found that all the other pistons experience a similar pressure, and that a corresponding force P must be applied to each of them, in order to retain them in their places. Thus the pressure communicated to the fluid has been transmitted in all directions, for all the pistons experience it, whatever their position, and whatever the shape of the vessel; and the transmitted pressure is equal to that communicated.



These results are the same, whatever the number of the pistons, and wherever placed. It follows, therefore, that every portion of the surface of the containing vessel, equal in area to that of the piston, experiences a pressure P. Thus, if the area of a piston is one square inch, and a pressure of 1 lb. is exerted upon it, a corresponding pressure of 1 lb. is experienced by every square inch of the surface of the containing vessel; and similarly with any other pressure.

196. If A and B, the areas of whose lower surfaces are a and b respectively, be two pistons fitted into the upper surface of a vessel filled with fluid, and if Q be the pressure experienced by B, when a pressure P is exerted upon A; then,

$$Q : P :: b : a.$$

For, in consequence of the pressure upon the piston A, a pressure P is transmitted to every portion of the surface of the containing vessel, whose area equals a, and therefore to every unit of area a pressure is transmitted equal to $P \div a$,

Since the lower surface of B contains b units of area, the whole

* For the reason of this provision, see Art. 207.

pressure experienced by B is $Pb \div a$. But, by the hypothesis, Q denotes this pressure, therefore $Q = Pb \div a$; that is,

$$Q : P :: b : a,$$

or,
$$\frac{\text{pressure on B}}{\text{pressure on A}} = \frac{\text{area of B}}{\text{area of A}}.$$

As an illustration, let the lower surface of the piston A be a circle whose diameter is 1 inch, and that of B a circle whose diameter is 1 foot, then

$$\frac{\text{area of B}}{\text{area of A}} = \frac{12^2}{1^2} = 144,$$

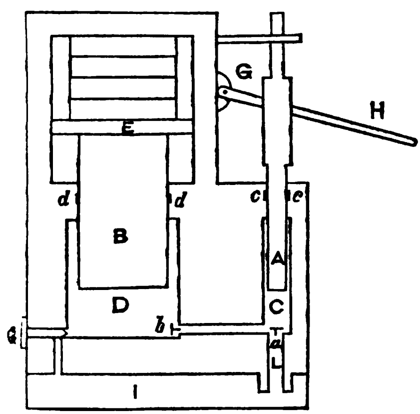
and, therefore,

$$\frac{\text{pressure on B}}{\text{pressure on A}} = 144,$$

or,
$$\text{pressure on B} = 144 \times \text{pressure on A};$$

so that if a weight of 1 lb. rest upon the piston A, a weight of 144 lbs. must be placed upon the piston B in order to maintain equilibrium; and conversely, if 144 lbs. press on the piston B, the equilibrium may be maintained by a pressure of 1 lb. on A.

197. Bramah Press. This powerful machine, invented by Mr. Bramah, is an interesting application to practical purposes of the characteristic property of fluids.



C and D are two cylindrical vessels, connected with each other by means of the pipe ab. A and B are two solid pistons working in water-tight collars cc, dd. The piston B, the diametero f which is much greater than that of A, supports a plate E, upon which the substance to be pressed is placed. A is capable of being worked up and down by a lever GH, having its

fulcrum at G. L is a pipe leading into a reservoir I, *a* is a valve opening upward, and *b* a valve opening into the vessel D.

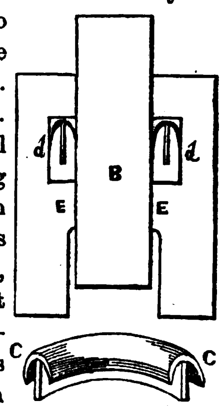
Let the vessel CD be supposed to be filled up with water, when the pistons are in the position represented in the figure. Let A be forced down with a pressure P, then, by the preceding Article, B is forced up with a pressure equal to

$$P \times \frac{\text{area of B}}{\text{area of A}}.$$

If B yield to this pressure, the piston A will descend, and a portion of the water in C will be forced into the vessel D. The return of the fluid will be prevented by the valve *b*. If the piston A be now raised, fresh water will be pumped up into the vessel C from the reservoir I, and if A be again forced down with a pressure P, B will be again forced up with the same pressure as before. This process may be continued as long as the substance yields to the pressure exerted upon it.

The pressure upon B may at any time be removed by unscrewing the plug *e*, by which the water is allowed to flow back into the reservoir.

198. In consequence of the immense pressure exerted by the Bramah Press, especial care is necessary to prevent the escape of the water between the piston and the cylinder in which it works. This is effected by the following contrivance. An annular piece of leather, after being well soaked, is formed into a collar, by doubling it over the rim of a metal cylindrical ring, in the manner shown at *c c*, which represents one-half of the collar and ring. The collar, along with the ring, is placed in a recess cut in the part of the cylinder in which the piston works, as shown at *d d*. When the press is at work, the water forces its way through



the crevice E E, and enters the recess *d d*. The pressure of the water, acting against the under surface of the leather, forces it on one side of the ring against the piston, and on the other side against the cylinder, and by this means the escape of the water is effectually prevented. In fact, the greater the pressure, the more firmly is the leather pressed against the piston and the walls of the cylinder; so that, by this ingenious contrivance, the greater the danger of leakage, the greater is the protection provided against it.

199. The safety valve is another important application of the same principle. A portion of the boiler of a steam-engine, whose area contains a square inches, is furnished with a valve opening outwards. The valve is so contrived as to admit of being loaded with certain weights. If a weight of aP lbs. be placed on the valve, it is pressed down with a pressure of P lbs. to the square inch. If then at any time the pressure of the steam be greater than P lbs. to the square inch, the valve will be forced open, and the steam will escape; and since the pressure of the steam is the same on every inch of the surface of the containing vessel, no portion of the boiler will experience a greater pressure than P lbs. to the square inch. So that if P be less than the pressure per square inch which the weakest portion of the boiler can bear, the boiler can never burst.

EXAMPLES.

1. If two circular pistons, whose diameters are in the ratio of 3 : 7, are inserted in the sides of a closed vessel filled with a weightless fluid, what pressure will be necessary to keep the smaller piston in its place when the larger piston is pressed in with a force of 245 lbs.?

The pressure required is 45 lbs.

2. Two circular pistons are inserted in the sides of a closed vessel, what must be the ratio of their diameters, in order that when the vessel is filled with any fluid, and a pressure of 9 oz. is applied to the smaller piston, a pressure of 64 lbs. may be transmitted to the larger?

The diameters must be in the ratio of 3 : 32.

3. A piston, whose area is 4 square inches, is inserted into one side of a cubical vessel filled with water; required the pressure upon the entire surface of the vessel when the piston is pressed in with a force of 2 lbs., the edge of the vessel being 15 inches.

The required pressure is 6 cwt. 1 lb.

4. The diameter of the head of a cylindrical cask is 20 inches, and the height of the cask is 30 inches; what is the total pressure upon the surface when a piston of one inch in diameter is pressed in with a force P?

The required pressure is 3199 P.

5. The diameter of the large piston in a Bramah press is 25 inches, of the small piston $\frac{1}{4}$ inch, the lever is 2 feet 6 inches long, and is attached to the small piston-rod at a point 2 inches from the fulcrum; what weight would be raised by a power of 1 cwt.?

The weight required is 7500 tons.

CHAPTER XIII.

ON THE EQUILIBRIUM OF FLUIDS ACTED UPON BY GRAVITY.

200. All the particles of a fluid, equally with those of any solid body, are subject to the action of gravity. Any portion, therefore, however small, of any fluid is drawn towards the earth with a certain degree of force, that is, is possessed of weight. Other forces, beside that of gravity, may act upon the particles of a fluid; and when that is the case, the determination of the conditions of equilibrium is difficult. We here confine ourselves to an examination of the circumstances which attend the equilibrium of fluids, when their particles are acted upon by the force of gravity alone.

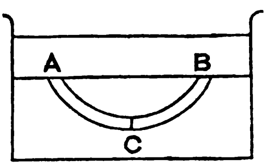
201. *The surface of a fluid at rest is horizontal.*

For, if possible, let any portion of the surface of a fluid at rest be not horizontal. There can then be taken in the surface three points which do not lie in the same horizontal plane, Let the fluid be supposed to be divided by a plane passing through these three points, and let the portion of fluid below and that above be supposed to become solid. The latter will then be a weight, unsupported by any force, at rest upon an inclined plane. But since there is no friction between the particles of a fluid, this is impossible. Therefore, no three points can be taken in the surface of a fluid at rest which do not lie in the same horizontal plane, and consequently the surface of a fluid at rest must be horizontal.

202. The total pressure of a fluid at rest, upon any surface in contact with it may be, and most commonly is, the sum of two different pressures; the one a pressure transmitted simply by the fluid, and the other a pressure arising from the weight of the fluid itself. The former, as we have seen, is the same upon every unit of a surface in contact with the fluid. The latter, it will be seen hereafter, varies with the position of the surface. When the latter is known in any case, it is only necessary to add to it the amount of transmitted pressure corresponding to the area of the surface, and the total pressure is found. If, then, we can determine the pressure of a fluid upon any surface arising from the action of gravity upon its particles, upon the supposition that no external pressure is communicated to the fluid, the problem is completely solved.

203. *The pressures of a fluid at rest at any two points in the same horizontal plane are equal.*

Let A and B be any two points in a fluid at rest, lying in the same horizontal plane. Let P be the pressure of the fluid upon a unit of area at A, and Q the pressure upon a unit of area at B; then shall P and Q be equal.



Let ACB be any portion of the fluid detached from the rest and enclosed in a rigid tube, the parts AC, BC being perfectly symmetrical. Let C be the lowest point of this tube, and let a vertical plate at C become rigid. The pressures on each side of this plate arising from the fluid in ABC are equal, and consequently, since the fluid is at rest, the pressure transmitted from A and B must be equal, that is, P and Q must be equal.

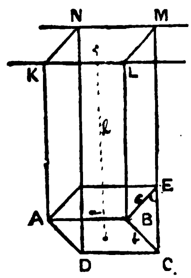
204. The pressure of a fluid at rest upon any plane must be in a direction perpendicular to the plane.

For the only force opposed to the pressure of the fluid on the

plane is the resistance of the plane itself, and since there is equilibrium, these two forces must be equal and opposite. The resistance of the plane is perpendicular to the plane; this, consequently, must be the direction of the pressure of the fluid.

205. *To find the pressure of a fluid upon a rectangular plane immersed in it, so that one of its edges is parallel to the surface of the fluid.*

Let ABCD be the given plane, having AB parallel to the surface of the fluid. Let the vertical lines through A, B, C, D, meet the surface in K, L, M, N. Let the portion of the fluid included within the prism CK be supposed to become solid. The forces acting upon CK are the pressures on its surfaces and its own weight, and these are in equilibrium. The pressures on the vertical faces being (Art. 204) horizontal, produce no effect in the vertical direction. The only forces acting in the vertical direction are the vertical component of the pressure on ABCD, and the weight of the fluid mass CK; and, since there is equilibrium, these must be equal and opposite.



Let AE be a horizontal section of the prism through AB. Let $AB = a$, $BC = b$, and $BE = c$. Let h be the distance of the centre of ABCD from the surface of the fluid, P the pressure of the fluid upon ABCD, and w the weight of a cubic unit of the fluid. The content of CK equals the area of the face NL multiplied by its mean depth, or h;

$$\therefore \qquad \text{content of CK} = ach,$$

$$\therefore \qquad \text{weight of CK} = wach.$$

By Art. 103, the vertical component of P is $P \cdot \frac{BE}{BC}$, or $P\frac{c}{b}$;

$$\therefore \qquad P\frac{c}{b} = wach,$$

$$\therefore \qquad P = wabh.$$

But $wabh$ is the weight of a column of the fluid whose base is ab, or the plane ABCD, and whose height is h; and the centre of ABCD is also its centre of gravity. Therefore the pressure on the given plane is the weight of a column of the fluid whose base is the given plane, and whose height is the depth of the centre of gravity of the plane below the surface of the fluid.

If the vertical lines through A, B, C, D, do not meet the surface of the fluid mass, let LN be any horizontal plane, some portion of which lies vertically under the surface of the fluid mass, and let h' be the depth of this plane below the surface. Then, by what has been just proved, the pressure on any unit of area in this plane vertically under the surface of the fluid is wh'. Therefore, by Art. 203, the pressure on every unit of plane NL is wh', and consequently the pressure on the area KLMN is $wach'$. Since this pressure acts vertically and concurrently with the weight of CN, the vertical component of P must be equal to the sum of these two forces;

$$\therefore \qquad P\frac{c}{b} = wach + wach',$$

$$\therefore \qquad P = wab\,(h + h');$$

and $h + h'$ is the entire depth of the centre of gravity of ABCD below the surface of the fluid. In like manner it may be shown, that in all cases the pressure of a fluid on any rectangular plane, having one of its sides parallel to the surface of the fluid, is the weight of a column of the fluid whose base equals the area of the plane, and whose height is the depth of the centre of gravity of the plane below the surface of the fluid.

(not resultant).

206. *To determine the total pressure of a fluid upon any surface.*

Let the surface be divided into small rectangular planes, each having one of its sides parallel to the surface of the fluid. Let A be the area of the whole plane, and z the distance of its centre of gravity below the surface of the fluid. Let A_1, A_2, A_3, &c. be the

areas of the several rectangles, and z_1, z_2, z_3, &c. the distances of their centres of gravity below the surface of the fluid; then,

$$\text{pressure on given surface} = wA_1z_1 + wA_2z_2 + wA_3z_3 + \&c.$$
$$= w(A_1z_1 + A_2z_2 + A_3z_3 + \&c.)$$

But (Art. 69) $\qquad Az = A_1z_1 + A_2z_2 + A_3z_3 + \&c.;$

$\therefore \quad$ pressure on given surface $= wAz$,

or the pressure on the surface is the weight of a column of the fluid whose base equals the area of the given surface, and whose height is equal to the depth of the centre of gravity of the surface below the surface of the fluid.

Ex. 1. To find the pressure on a rectangular plane 10 in. by 4, when immersed in water so that its centre of gravity is at a depth of 20 inches.

The area of the plane is 10×4, or 40 square inches. Hence the pressure on the plane is the weight of 40×20, or 800 cubic inches of water.

The weight of a cubic foot of pure water is 1000 oz. avoirdupois, and therefore the weight of a cubic inch is $1000 \div 1728$ oz.

$$\therefore \qquad \text{pressure required} = \frac{800 \times 1000}{1728}$$
$$= 462 \cdot 96 \text{ oz.}$$

Ex. 2. To compare the pressures on the bottom and side of a cubical vessel filled with any fluid.

Let a be the length of the edge of the cube, and w the weight of a cubic unit of the fluid. The area of the bottom of the vessel is a^2, and the depth of its centre of gravity is a, therefore the pressure on the bottom is equal to wa^3.

The area of the side of the vessel is a^2, and the depth of its centre of gravity is $\frac{1}{2}a$, therefore the pressure on the side is equal to $\frac{1}{2}wa^3$. Hence the pressure on the side of the vessel is one-half of that on the bottom.

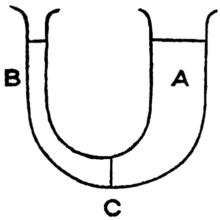
207. If the student will now revert to Art. 195, he will see why it was necessary to insert the provision, that the pistons be maintained in their position by some mechanical contrivance. For if the pistons (as in the case with the pistons C, D) be situated anywhere except upon the uppermost surface of the vessel, they will experience a pressure greater or less according to their distance below the surface of the fluid, and will, in consequence of this pressure, be forced out, if not prevented.

The pistons A and B, which press upon the uppermost surface of the fluid, experience no pressure from the weight of the fluid itself; whatever force, therefore, is impressed upon them is transmitted undiminished to every part of the fluid. But if a pressure be applied elsewhere, at C for instance, and no other force act upon the piston, then a part of this pressure will be employed in counterbalancing the pressure of the fluid upon C, and the remainder only will be the pressure transmitted by the fluid. Hence, if a be the area of the piston, h the distance of its centre of gravity below the surface of the fluid, and w the weight of a cubic unit of the fluid, and if $a\mathrm{P}$ be the pressure exerted upon the piston, the pressure transmitted to every corresponding area is $a\mathrm{P} - wah$, or the pressure transmitted to every unit of area is $\mathrm{P} - wh$.

208. Since the pressure of a given fluid upon a given horizontal plane depends only upon the depth of the plane beneath the surface of the fluid, it follows that, if the bottoms of any number of vessels of various shapes be of equal area, and the same fluid be poured to the same height in all, the pressure upon the bottom of each vessel will be the same, however different the quantity of fluid in each may be. If the bottoms of the vessels be moveable, it will be found by experiment that the same force is necessary in each case to retain them in their respective positions.
This is sometimes called the Hydrostatic Paradox.

209. *If a fluid be poured into any one of a number of open vessels having a free communication with each other, the fluid will rise to the same height in all.*

Let A and B be any two such vessels, having the lowest point at C.



Let a thin vertical plate of the fluid passing through C become rigid. Let a be the area of this plate, h the distance of its centre of gravity below the surface of the fluid in A, and h' its distance below the surface in B.

Let w be the weight of a cubic unit of the fluid. Then the pressure exerted on the one side of the plate by the fluid in A is wah, and that exerted on the other side by the fluid in B is wah'. But since there is equilibrium, these pressures must be equal; therefore

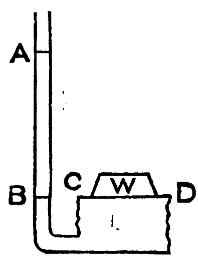
$$wah = wah',$$

or,

$$h = h'.$$

210. Hydrostatic bellows. AB is a narrow tube communicating with a vessel formed by uniting together two pieces of wood by some flexible and waterproof substance. If water be poured down the tube, it will enter the vessel, and raise a large weight W placed upon CD.

When equilibrium exists, let A be the surface of the water in the tube, and let B lie in the same horizontal plane with CD. Let w be the weight of a cubic unit of water, and let h be the depth of B or CD below A.



The pressure upon each unit of area in CD is wh. Therefore, if b = number of units in the area of CD, the total pressure on CD = wbh; and, since there is equilibrium, this must equal the weight supported; therefore

$$W = wbh.$$

If a = area of the horizontal section of the pipe at B, weight of fluid in AB = wah.

$$\therefore \quad W = \frac{b}{a} \times \text{weight of fluid in AB.}$$

If CD and the tube are circular, and if R be the radius of CD, and r the radius of horizontal section of the tube,

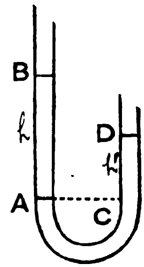
$$b : a :: R^2 : r^2,$$

$$\therefore \quad W = \left(\frac{R}{r}\right)^2 \times \text{weight of fluid in AB.}$$

211. *To determine the position of equilibrium when two fluids which do not mix meet in a bent tube.*

Let A be the common surface to the two fluids. Let B be the surface of the one fluid, and D that of the other, when there is equilibrium.

Let h be the distance of B above the horizontal plane at A, and h' the distance of D above the same plane.



Let w be the weight of a cubic unit of the fluid in AB, and w' of the fluid in ACD. Let a be the area of the horizontal section of the tube at A. If a thin plate of the fluid at A be supposed to become solid, since there is equilibrium, the pressures on its upper and under surfaces must be equal. The pressure on its upper surface is wah, and on its lower surface $w'ah'$; therefore

$$wah = w'ah',$$

$$wh = w'h';$$

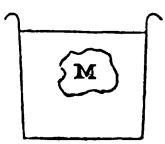
$$\therefore \quad \frac{h}{h'} = \frac{w'}{w}.$$

212. The pressure of a fluid at any point of a surface acts in the direction of the normal to the surface at that point. The pressure found in Art. 206 is the *total pressure*, or the sum of the pressures at all points of the surface. The resultant of any number of forces is equal to their sum only when the forces are parallel, and hence, the total pressure already found will be the *resultant pressure* only when all the normals are parallel, that is, when the surface is a plane. If the surface be not a plane, the

resultant pressure will not be the same as the total pressure, but less than it, since the resultant of any number of forces which are inclined to each other is less than their sum. The determination of the resultant pressure requires in general a knowledge of the differential calculus. There are, however, certain cases in which the resultant pressure may be more easily found. These may be classified under the following two divisions: first, those in which the given surface is the entire surface in contact with the fluid, when the body is either wholly or partially immersed in the fluid; and secondly, those in which the given surface is bounded by one or more *plane* curves.

213. *To find the resultant pressure of a fluid on the surface of a solid, wholly or partially immersed in it.*

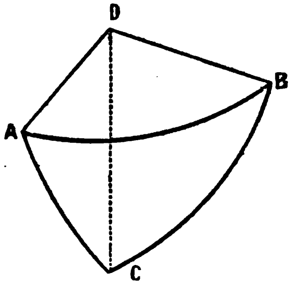
Let any portion M of any fluid at rest become solid, and since the equilibrium is not thereby disturbed, the forces acting upon M must be in equilibrium. The only forces acting upon M are the weight of M acting vertically downwards at the centre of gravity of M, and the pressures of the fluid upon the surface of M. Hence, the resultant of these pressures must be equal and opposite to the weight of M acting at its centre of gravity. But the fluid will exert the same pressure upon the surface of any other body of the same form as M occupying its place. Hence, the resultant of the pressure of a fluid on the surface of a solid immersed in it is equal to the weight of the fluid displaced, and acts vertically upwards through the centre of gravity of the fluid displaced.



214. When the given surface is not the entire surface in contact with the fluid, the resultant pressure may be found by the following method, if the surface be bounded by plane curves.

Let ABC be the given surface, bound by the plane curves AB, BC, and CA. Let planes passing through these curves intersect in AD, DB, and DC, and let ADBC be a portion of the fluid mass,

bounded by these planes and the given surface. Let this portion of the fluid be supposed to become solid. It is then a body at rest under the action of the following forces:—the weight of the fluid mass, the pressures on the planes, and the pressure on the given surface. The last, therefore, must be equal and opposite to the resultant of the rest. By finding then the weight of the fluid mass, and compounding it with the pressures on the plane surfaces, we obtain the required resultant pressure on the given surface.



In applying this method, it will often be convenient to find the required pressure by finding separately its horizontal and vertical components; the former being equal to the resultant of the other horizontal pressures acting on ABCD, and the latter equal to the resultant of the other vertical pressures.

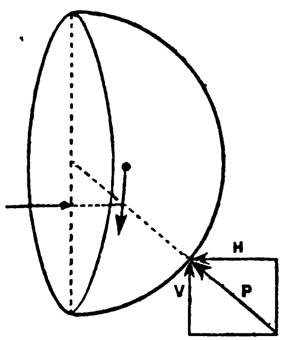
Ex. 1. A sphere is immersed to a given depth in a given fluid, to find the magnitude and direction of the resultant pressure on a hemisphere whose bounding plane is vertical.

Let r = radius of sphere, h = depth of the centre, and w = the weight of a unit of the fluid, then the pressure on the bounding plane is equal to

$$w\pi r^2 h,$$

and acts horizontally; and the weight of the fluid included between the hemisphere and its bounding plane is equal to

$$\tfrac{2}{3} w\pi r^3,$$



and acts vertically. These two forces are in equilibrium with the

required pressure (P). Therefore, if V be the vertical and H the horizontal components of P,

$$V = \tfrac{2}{3} w \pi r^3,$$

and $$H = w \pi r^2 h.$$

Therefore, $$P = \sqrt{(V^2 + H^2)},$$
$$= \tfrac{1}{3} w \pi r^2 \sqrt{(4r^2 + 9h^2)};$$

and if θ be the inclination of P to the vertical line,

$$\tan \theta = \frac{H}{V} = \frac{3h}{2r}.$$

COR. If the sphere be just immersed, then $h = r$, and resultant pressure $= \frac{1}{3} w \pi r^3 \sqrt{13}$.

Ex. 2. A sphere is immersed to a given depth in a fluid to find the resultant on either of the four parts into which the surface is divided by two vertical planes passing through the centre, and at right angles to each other.

Let r = radius of sphere, h the depth of the centre, and w the weight of a unit of the fluid.

The pressure on each of the bounding planes is equal to

$$\tfrac{1}{2} w \pi r^2 h.$$

These pressures are both horizontal, and at right angles to each other, therefore the resultant horizontal pressure is

$$\tfrac{1}{2} \sqrt{2} \,.\, w \pi r^2 h.$$

The weight of the fluid included between the given surface and the bounding planes is

$$\tfrac{1}{3} w \pi r^3.$$

Hence, as before,

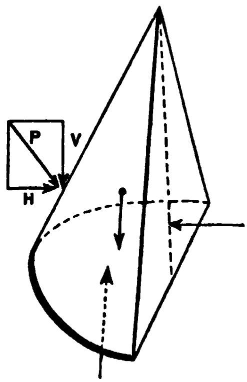
$$V = \tfrac{1}{3} w \pi r^3,$$
$$H = \tfrac{1}{2} \sqrt{2} \,.\, w \pi r^2 h,$$

therefore, $$P = \tfrac{1}{6} w \pi r^2 \sqrt{(4r^2 + 18h^2)}.$$

COR. If the sphere be just immersed, then

resultant pressure $= \frac{1}{6} w \pi r^3 \sqrt{22}$.

Ex. 3. A cone immersed in a fluid, with its vertex on the surface and its axis vertical, is bisected by a plane passing through the axis, to find the resultant pressure on either half of the conical surface.



Let h = height of the cone, r = radius of base, and w the weight of a unit of the fluid. Then the pressure on the vertical plane is equal to

$$\tfrac{2}{3} w r h^2,$$

and acts horizontally. The pressure on the base is

$$\tfrac{1}{2} w \pi r^2 h,$$

and acts vertically upwards, and the weight of the included fluid is

$$\tfrac{1}{6} w \pi r^2 h.$$

Hence,
$$V = \tfrac{1}{2} w \pi r^2 h - \tfrac{1}{6} w \pi r^2 h,$$
$$= \tfrac{1}{3} w \pi r^2 h,$$
and
$$H = \tfrac{2}{3} w r h^2.$$
$$\therefore \quad P = \tfrac{1}{3} w r h \sqrt{(4h^2 + \pi^2 r^2)}.$$

215. If any body hang freely by a cord, we have seen (Art. 60) that the centre of gravity lies in the vertical line drawn through the point of suspension, that is, in the line of the cord; and the tension in the cord equals the weight of the body.

If such a body be wholly immersed in a fluid, it has been shown in Article 213, that it will be pressed upwards by a force equal to the weight of the fluid displaced, which in this case is the weight of a portion of the fluid equal to the bulk of the body; and this pressure is directly opposite to the weight of the body; therefore the tension in the cord will be the weight of the body diminished by the weight of an equal bulk of the fluid.

216. *To determine the conditions of equilibrium of a floating body.*

If a body floating in a fluid be at rest, the forces acting upon it are the weight of the body, which acts as its centre of gravity, and the resultant pressure of the fluid which acts at the centre of gravity of the fluid displaced. ie. center of buoyancy or displacement.

The necessary conditions of equilibrium are, that these forces be *equal* and *opposite*. The two conditions therefore are,

First, That the weight of the body be equal to that of the fluid displaced; and,

Secondly, That the centres of gravity of the body and the fluid displaced be in the same vertical line.

As a body cannot displace a quantity of fluid greater than its own bulk, if the weight of the body be greater than that of an equal bulk of the fluid, the body cannot float in that fluid. If the weight of the body be exactly equal to that of an equal bulk of the fluid, the body will have in no position any tendency either to rise or to sink, provided only it be entirely covered by the fluid.

EXAMPLES.

1. To what depth must a surface be sunk in water that the pressure upon it may be at the rate of 25 lbs. to the square inch?

The required depth is $57\frac{3}{5}$ feet.

2. If the side of a vessel be a triangle, having its base at the bottom of the vessel, show that the pressure exerted upon it by any fluid when the vessel is full is to that exerted when the vessel is filled to only half its depth, as 16 : 5.

3. If in the preceding the base of the triangle be the top of the vessel, show that the pressure when the vessel is full is to the pressure when the vessel is only half filled, as 8 : 1.

4. Show generally, in the last example, that the pressure on the side when the vessel is full is to the pressure when the vessel is filled to one nth of its depth, as $n^3 : 1$.

5. Find the total pressure upon the surface of a globe, 2 feet in diameter, when immersed in water so as to be just covered.

The total pressure is 785·39 lbs.

6. A globe, 2 feet in diameter, when floating in water, is half immersed, what is its weight?

The weight of the globe is 130·9 lbs.

7. A conical body floats in water with its vertex downwards, what will be the depth of the vertex below the surface of the fluid, the weight of the cone being 200 oz., its height 24 inches, and the diameter of its base 12 inches?

The required depth is $\sqrt[3]{(3{\cdot}05)}$ feet.

8. A cylindrical vessel is filled with water; compare the pressures upon the bottom and sides.

Let r be the radius of the base, and h the height of the vessel; then the pressure upon the bottom is to the pressure upon the sides as $r : h$.

9. A cylindrical vessel is filled half with water and half with a fluid twice as heavy as water; the two fluids do not mix; compare the pressures upon the upper and lower half of the sides.

The pressure upon the lower half is 4 times that upon the upper half.

10. A sphere, whose radius is 6 inches and weight 35 lbs., is suspended by a string; required the tension in the string when the sphere is wholly immersed in water.

The required tension is 2 lbs. 4·4 oz.

11. A rectangular plate, ABCD, hangs by a cord at A with its plane vertical, and is partly immersed in water, in the position of equilibrium the diagonal BD coincides with the surface of the water; required the weight of the plate and the tension in the string.

Let a and b be the sides of the plate in inches, and c its thickness; then, if w be the weight of a cubic inch of water, the weight of the plate is equal to

$$\frac{2\,wabc}{3},$$

and the tension in the string is equal to

$$\frac{wabc}{6}.$$

12. If a triangular plate ABC, having its sides AB, AC unequal, be suspended at A, and be partially immersed in any fluid, show that in the position of equilibrium the line drawn from A to the bisection of the opposite side cannot be vertical.

13. If any plate be suspended, and partially immersed in any fluid then in the position of equilibrium, the moment of the volume of the displaced fluid about the point of suspension is to the moment of the volume of the plate as the weight of the plate is to the weight of an equal volume of the fluid.

Let V and V′ be the volumes of the entire plate, and of the immersed part, and q, q', the distances of their centres of gravity from the vertical line through the point of suspension. Let W be the weight of the plate, and w the weight of a unit of the fluid; then the upward thrust of the fluid is equal to

$$wV'.$$

The resultant of the weight of the plate and the thrust of the fluid must pass through the point of suspension; therefore, taking the moments about this point, we have

$$wV'q' = Wq.$$

Whence, $$\frac{V'q'}{Vq} = \frac{W}{wV}.$$

14. In the preceding, if V', q', apply not to the part immersed, but to the part not immersed, show that

$$\frac{V'q'}{Vq} = 1 - \frac{W}{wV}.$$

15. A triangular plate ABC, having the angle at B a right angle, is suspended at A, and partially immersed in a fluid; required the height of A above the surface of the fluid, in order that in the position of equilibrium the side AB may be vertical.

Let $AB = a$; let W be the weight of the plate, and W′ the weight of an equal volume of the fluid; the required distance is equal to

$$a\sqrt[3]{\left(1 - \frac{W}{W'}\right)}. \quad ? \quad a\sqrt[2]{\left(1-\frac{W}{W'}\right)}$$

16. A conical vessel having its vertex downwards, and filled with a given fluid, is bisected by a vertical plane passing through the axis, to find the resultant pressure on either portion of the surface.

Let h be the height of the cone, r the radius of its base, and w the weight of a unit of the fluid, then the required pressure is equal to

$$\tfrac{1}{6}wrh\sqrt{(4h^2 + \pi^2 r^2)}.$$

17. In the preceding, if the cone be divided into four equal portions by two planes through the axes at right angles to each other, show that the resultant pressure on each portion of the conical surface is equal to

$$\tfrac{1}{12}wrh\sqrt{(8h^2 + \pi^2 r^2)}.$$

CHAPTER XIV.

ON SPECIFIC GRAVITY.

217. The specific gravity of any substance is the ratio of the weights of equal volumes of that substance, and of a certain standard substance.

The standard for solids and liquids is distilled water, at a temperature of 60°.

The standard for gases is pure atmospheric air, at a temperature of 60°, with the barometer at 30 inches.

Hence, if the specific gravity of any solid or liquid be 2, the weight of a cubic inch of the substance is twice the weight of a cubic inch of water. Or generally, if s be the specific gravity of any substance, and w the weight of a cubic unit of the standard, the weight of a corresponding unit of the substance is sw; and if V be the volume of the substance, and W its weight,

$$W = Vsw.$$

=62½ lbs. A cubic foot of pure water, at a temperature of 60°, weighs 1000 ounces avoirdupois; and 100 cubic inches of air, at the standard temperature and pressure, weighs 31 grains. and 7000 gr in 1 lb. av.

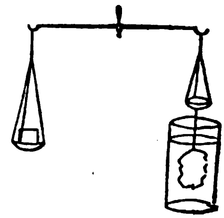
The absolute weight of any substance may hence be easily found from a table of specific gravities.

For example, to find the weight of a cubical block of marble, whose side is 4 feet, the specific gravity of the marble being 2·7.

The block contains 4^3, or 64 cubic feet of marble. Therefore,

$$\begin{aligned} \text{weight of the block} &= 64 \times 2.7 \times 1000 \text{ oz.} \\ &= 10800 \text{ lbs.} \\ &= 4 \text{ tons, } 16 \text{ cwt., } 1 \text{ qr., } 20 \text{ lbs.} \end{aligned}$$

218. The hydrostatic balance is, in its simplest form, a common pair of scales, with a fine thread attached to the under surface of one of the scale-pans. By this arrangement, any solid substance can be weighed, both in the ordinary way, and also by means of the thread when immersed in any fluid.



219. *To find the specific gravity of a body heavier than water.*

Weigh the body both in air and in water. The loss of weight in water is, as seen in Art. 215, equal to the weight of the water displaced, that is, since the body is entirely immersed, of a volume of water exactly equal to that of the given body. Therefore,

Let weight in air = w_a. — water = w_e

$$\text{sp. gr.} = \frac{\text{weight of the body}}{\text{loss of weight in water}}.$$

sp. gr. = $\frac{w_a}{w_a - w_e}$

Let weight in air = a — water = b. sp. gr. = $\frac{a}{a-b}$.

220. *To find the specific gravity of a body lighter than water.*

To the given body attach some other body heavy enough to sink it in water; weigh the two together both in air and in water; the loss of weight is equal to the weight of the water displaced by both. Then weigh the heavy body in air and in water; the loss of weight is equal to the weight of the water displaced by the heavy body.

The difference of the two losses is therefore equal to the weight of water displaced by the given body. Hence,

$$\text{sp. gr.} = \frac{\text{weight of body}}{\text{difference of the two losses}}.$$

221. *To find a relation between the weights and specific gravities of the components, and the weight and specific gravity of the compound.*

Let W_1, W_2, &c. be the weights; V_1, V_2, &c. the volumes; and s_1, s_2, &c. the specific gravities of the components. Let W be the weight, V the volume, and s the specific gravity of the compound.

Art. 220. Let light weight in air = w_a — and so... heavy weight weight in air = w_h

Total w. in air = $w_h + w_a$ } ∴ weight of water displaced by both weights = $w_h + w_a - w_E$
— water = w_E }
— in air = w_h } ∴ — — — — — heavy weight = $w_h - w_e$
— water = w_e }

Look next page

Then if no change of volume result from the composition,

$$V = V_1 + V_2 + \&c.$$

But by Art. 217, $V = \dfrac{W}{sw}$; $V_1 = \dfrac{W_1}{s_1 w}$, and so on. Therefore

$$\frac{W}{sw} = \frac{W_1}{s_1 w} + \frac{W_2}{s_2 w} + \&c.$$

$$\therefore \qquad \frac{W}{s} = \frac{W_1}{s_1} + \frac{W_2}{s_2} + \&c.$$

222. If the volume be changed by the composition:—let the volume after composition be to the sum of the volumes of the components in the ratio of $m : n$. Then

$$V = \frac{m}{n}(V_1 + V_2 + \&c.)$$

$$\therefore \qquad \frac{W}{s} = \frac{m}{n}\left(\frac{W_1}{s_1} + \frac{W_2}{s_2} + \&c.\right)$$

223. *To find a relation between the volumes and specific gravities of the components, and the volume and specific gravity of the compound.*

Using the same notation as in Art. 221,

$$W = W_1 + W_2 + \&c.$$

But, Art. 217, $W = Vsw$; $W_1 = V_1 s_1 w$, and so on. Therefore

$$Vsw = V_1 s_1 w + V_2 s_2 w + \&c.$$

$$\therefore \qquad Vs = V_1 s_1 + V_2 s_2 + \&c.$$

224. *In an alloy compounded of two known metals, to determine the proportion in which either of the metals enters.*

Let the alloy be compounded of the metals A and B, whose specific gravities are s_1 and s_2 respectively, and let s be the sp. gr. of the alloy. Let x be the weight of the quantity of the metal A contained in a mass of the alloy whose weight is X. Then the

∴ wt. of water displaced by light weight = $w_a + w_e - w_E$

∴ sp. gr. = $\dfrac{w_a}{w_a + w_e - w_E}$

weight of the metal B contained in the same mass is $X - x$. Hence (Art. 221)

$$\frac{X}{s} = \frac{x}{s_1} + \frac{X - x}{s_2};$$

$$\therefore \quad x\left(\frac{1}{s_1} - \frac{1}{s_2}\right) = X\left(\frac{1}{s} - \frac{1}{s_2}\right)$$

$$\frac{x}{X} = \frac{s_1}{s} \cdot \frac{s - s_2}{s_1 - s_2}.$$

Or, if it be required to determine the proportion with respect to volume, let y be the volume of the metal A contained in a volume Y of the alloy. Then $Y - y$ is the volume of the metal B contained in the alloy. Therefore (Art. 223),

$$Ys = ys_1 + (Y - y)s_2,$$

$$\therefore \quad y(s_1 - s_2) = Y(s - s_2);$$

$$\therefore \quad \frac{y}{Y} = \frac{s - s_2}{s_1 - s_2}.$$

225. *To determine the specific gravity of a fluid by means of the specific gravity bottle.*

The specific gravity bottle is simply a small flask fitted with a ground stopper.

Let w be the weight of the flask, x its weight when filled with the given fluid, and y its weight when filled with distilled water. Then $x - w$ is the weight of the given fluid contained in the flask, and $y - w$ the weight of a similar volume of distilled water.

$$\therefore \quad \text{sp. gr. of the fluid} = \frac{x - w}{y - w}.$$

Specific gravity bottles are very frequently made so as to contain exactly 1000 grains of pure distilled water. Then, if x and w are known in grains, Then $y = w + 1000$ $\therefore y - w = 1000$

$$\text{sp. gr. of the fluid} = \frac{x - w}{1000}.$$

226. *To determine the specific gravity of a fluid by weighing a solid body in it.*

Let w be the weight of the solid body, x its weight when suspended in the given fluid, and y its weight when suspended in distilled water. Then (Art. 215) $w-x$ is the weight of a quantity of the given fluid equal in volume to the solid body, and $w-y$ is the weight of a similar volume of distilled water;

$$\therefore \text{ sp. gr. of the fluid} = \frac{w-x}{w-y} = \frac{\text{loss of weight in the fluid}}{\text{loss of weight in water}}.$$

Or, if s be the specific gravity of the solid, then since (Art. 186)

Let w_s = wt. of given solid mass
— w_f = — — given fluid
— w_w = — — distilled water.
$w_s - w_f$ = — fluid displaced by solid

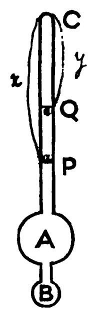
$$s = \frac{w}{w-y}.$$

$$\therefore \quad \text{sp. gr. of the given fluid} = s.\frac{w-x}{w}.$$

$w_s - w_w$ = wt. of dist water displaced by solid
∴ sp. gr. of fluid $= \dfrac{w_s - w_f}{w_s - w_w}$; Let $s = \dfrac{w_s}{w_s - w_w}$
∴ — — $= s.\dfrac{w_s - w_f}{w_s}$.

227. *To determine the specific gravity of a fluid by means of the common hydrometer.*

The common hydrometer consists of a small hollow sphere A, to which there is attached on one side a slender graduated stem, and on the other a smaller sphere B, made of such a weight as to allow the whole instrument to float with the sphere A entirely immersed.



A table accompanies the instrument, in which are given the specific gravities corresponding to the several divisions of the stem. The specific gravity of any fluid is found by observing the degree to which the instrument sinks when placed in it, and then referring to the tables.

To explain the construction of the tables; let the instrument sink to P when placed in distilled water, and to Q when in some other fluid whose specific gravity, say, is s. Let V be the volume of the hydrometer, and a the area of the section of the stem. Let CP $= x$, and CQ $= y$, and let w be the weight of a unit of water.

$(V - ax)w$ = ~~weight~~ of displaced water = wt. of hydrometer
$(V - ay)sw$ = — — — fluid = — —

$$\therefore \frac{(V-ax)}{(V-ay)} = s$$

By Art. 216, when a body floats, the weight of the body is equal to that of the fluid displaced.

The quantity of water displaced is equal to the volume of the hydrometer, minus the part CP, that is, is equal to $V - ax$. Therefore the weight of the displaced water is equal to

$$(V - ax)\,w.$$

Similarly the volume of the fluid displaced is $V - ay$, and therefore the weight of the fluid displaced is equal to

$$(V - ay)\,sw.$$

But the weights of the displaced water and the displaced fluid are equal, since each is equal to the weight of the instrument. Therefore,

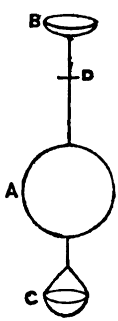
$$(V - ay)\,sw = (V - ax)\,w.$$

$$\therefore \quad s = \frac{V - ax}{V - ay}.$$

The tables are then formed by calculating the value of this fraction for different values of y. Vide Note on next page.

228. *To determine the specific gravity of a fluid by means of Nicholson's hydrometer.*

Nicholson's hydrometer consists of a hollow sphere, or some other symmetrical figure, connected to a stem which carries a cup at each end. The lower cup is loaded, so as to secure stability of equilibrium when the instrument floats with the stem vertical.



When in use, the instrument is sunk to an invariable depth, by means of a weight placed in the upper cup; the volume of fluid displaced is therefore always the same.

Let W be the weight of the instrument, and w the weight which must be placed in B to sink the instrument to the point D in distilled water; then the weight of water displaced is equal to

Art. 228

$$W + w.$$

Let W_h = wt. of hydr., and W_w = wt. req^d to sink hydr. to pt. D in water
W_f = -- given fluid
Then $W_h + W_w$ = wt. of water displaced by immersed part of hydr.
and $W_h + W_f$ = —— of fluid ——
-ersed part of the fluid is const. ∴ Sp. gr. of fluid = $\frac{W_h + W_f}{W_h + W_w}$.

Art. 229. Let W = weight of hydrometer.
W_w = wt. reqd. to sink hyd. to pt. D in standard fluid
— W_{p_1} = - - - - - - - - - - when W_s is in upper cup
— W_{p_2} = - - - - - - - - - - lower —
— W_s = Wt. of given solid in air

$W+W_w$ = Wt. of water displaced by hydr.
$W+W_{p_1}+W_s$ = — — — — } ∴ $W_s = W_w - W_{p_1}$
$(W+W_{p_2})+W_s$ = — — — by hydr. & solid together
∴ $W+W_w-W_{p_1}+W_{p_2}$ = - - - - - -
∴ $W_{p_2}-W_{p_1}$ = - - - by solid alone

∴ sp. gr. of solid = $\frac{W_w-W_{p_1}}{W_{p_2}-W_{p_1}}$

Let w' be the weight required to sink the instrument to D when placed in any given fluid, then the weight of the fluid displaced is equal to

$$W + w'.$$

The volumes displaced in each case are the same, therefore

$$\text{sp. gr. of the fluid} = \frac{W + w'}{W + w}.$$

229. *To find the specific gravity of a solid by means of Nicholson's hydrometer.*

Let W be the weight of the instrument, and w the weight required to sink it to D in the standard fluid; then the weight of water displaced by the instrument is

$$W + w.$$

Let the given solid be placed in the upper cup, and let w_1 be the weight which sinks the instrument to D; then the weight of the solid is equal to Then $w_1 + x = w$

$$w - w_1.$$

Also $W + w - w_1$ = weight of instr. & solid together, when solid is in upper cup.

Let the given solid be placed in the lower cup, and let w_2 be the weight which sinks the instrument to D; then the weight of the water displaced by the instrument and the solid together is equal to

And $W + w - w_1 + w_2$ = wt. of water & solid together when solid is in water

$$W + w - w_1 + w_2.$$

Hence, the weight of water displaced by the solid alone is equal to

$$w_2 - w_1.$$

Consequently, the specific gravity of the solid is equal to

$$\frac{w - w_1}{w_2 - w_1}.$$

Art. 222. Let v = vol. of immersed part of com. hydrom.; let ∂ = density i.e. wt. per unit vol. of liquid in wh. hydr. is immersed.
Let v_1 ∂_1 be the corresponding values for another liquid.
Then $v\partial$ = wt. of liquid displaced = wt. of hydr. = $v_1 d_1$ ∴ $d : d_1 = \frac{1}{v} : \frac{1}{v_1}$ and $v : v_1 = \frac{1}{d} : \frac{1}{d_1}$
Let d_1 ∂_2 ∂_3 be a series of densities in diminishing order, & V_1 V_2 V_3 the corresponding volumes immersed which will be in ascending order; then $\partial_1 : \partial_2 : \partial_3$: &c. = $\frac{1}{v_1} : \frac{1}{v_2} : \frac{1}{v_3}$: &c. & $V_1 : V_2 : V_3$: &c. = $\frac{1}{\partial_1} : \frac{1}{\partial_2} : \frac{1}{\partial_3}$ &c.
Also $\partial_1 : 2\partial_1 : 3\partial_1$ &c. = $\frac{1}{v_1} : \frac{2}{v_1} : \frac{3}{v_1}$: &c. N ∴ If the divns. of the scale of a common hydr. indicate equal differences of density, they must be so placed that the corresponding volumes immersed form a H. P

TABLE OF SPECIFIC GRAVITIES.

Platinum	21·47	Porcelain ....2·15 to	2·38
Gold	19·30	Brick	2·00
Mercury (congealed)	15·61	Ivory	1·82
" (fluent)	13·60	Phosphorous	1·77
Lead	11·38	Sugar	1·61
Silver	10·50	Lignum Vitæ	1·33
Bismuth	9·88	Box (Dutch)	1·33
Copper	8·67	Ebony (Indian)	1·21
Brass............7·82 to	8·40	Mahogany ....·64 to	1·06
Steel	7·83	Blood (human)	1·05
Iron (forged)	7·79	Milk	1·03
Tin	7·29	Sea Water	1·03
Iron (cast)	7·25	Sodium	·93
Zinc (compressed)	7·19	Proof Spirit	·92
" (common)	6·90	Box (French)	·91
Fluor Spar ......3·09 to	3·79	Potassium	·87
Quartz ....2·62 to	3·75	Beech	·85
Porphyry ......2·5 to	3·00	Alcohol	·80
Granite..........2·61 to	2·96	Plum Tree	·78
Plate Glass	2·94	Maple	·75
Rock (crystal)....2.58 to	2·89	Scotch Fir	·69
Marble	2·71	Cedar	·60
Green Glass............	2·64	Spruce Fir	·52
Portland Stone	2·50	Cork	·24

EXAMPLES.

1. What is the weight of a block of gold 12 inches long, 8 wide, and 6 deep?

The weight is 3 cwt. 66 lbs. $1\frac{3}{4}$ oz.

2. What is the weight of a silver globe, whose radius is 3 inches? The weight is 687·22 oz. avoir.

3. A block of granite weighs $7\frac{1}{2}$ cwt.; determine its volume in cubic feet, the specific gravity of the granite being 2·8.

The block contains 4·8 cubic feet.

4. A body weighs in water 3 oz. less than in air, what is its volume?

The required volume is 5·184 cubic inches.

5. A conical piece of cork, whose height is 12 inches, and specific gravity ·216, floats in water with its vertex downwards; what will be the depth of the vertex below the surface of the fluid?

The required depth is 7·2 inches.

6. If a conical body, whose height is h, and specific gravity s, float in a liquid whose specific gravity is s'; show that the depth of the vertex below the surface is $h \cdot \sqrt[3]{\left(\frac{s}{s'}\right)}$.

7. A cone, 10 inches in height, when floating in water with its base downwards, is immersed to the depth of 6 inches; to what depth will it be immersed when floating in a fluid whose specific gravity is ·943?

The required depth is 8 inches.

8. If 26 oz. of copper lose 3 oz. in water, and 7 oz. of zinc lose 1 oz., and if an alloy of copper and zinc, weighing 80 oz. lose 10 oz., what is the proportion of copper in the alloy?

The copper is to the zinc in the ratio of 13 : 7.

9. Two substances, A and B, whose specific gravities are s_1 and s_2, balance each other *in water;* determine the ratio of the weights of A and B.

The weight of A : the weight of B : : $s_1(s_2 - 1) : s_2(s_1 - 1)$.

10. The specific gravity of some adulterated milk is found to be 1·024; what proportion of water does it contain?

There is one pint of water in every 5 of the mixture.

11. Determine the greatest weight of Portland stone which can be floated by 100 cubic inches of cork.

The required weight is 4 lbs. 9·3 oz.

12. A crown, composed of gold and silver, and weighing 4 lbs. avoirdupois, displaces, when entirely immersed, 7 cubic inches, a lump of gold of the same weight displaces $5\frac{3}{4}$ inches, and a lump of silver of the same weight displaces $10\frac{1}{2}$ inches; determine the weight of gold contained in the crown.

The crown contains 2 lbs. $15\frac{3}{19}$ oz. avoirdupois of gold.

13. A Nicholson's hydrometer, weighing 200 grains, requires a weight of 66 grains to sink it to the standard depth in water, and a weight of 44 grains to sink it to the same depth in spirits of wine; determine the specific gravity of the spirits of wine.

The required specific gravity is ·917.

14. The specific gravity of a mixture of a pint of sulphuric acid and a pint of water is 1·71, the sp. gr. of the sulphuric acid being 1·85; determine the loss of volume resulting from the mixture.

The loss of volume is $\frac{1}{3}$ pint, or 11·52 cubic inches.

15. A body weighing 450 grains loses 210 grains in water and 150 grains in spirit, find the sp. gr. and volume of the body, and the sp. gr. of the spirit.

The sp. gr. of the body is 2·143; the volume of the body is ·829 cubic inches; and the sp. gr. of the spirit is ·714.

16. A substance placed in the upper cup of a Nicholson's hydrometer requires 5 grains to sink it to the standard depth in a fluid whose specific gravity is ·918, and 50 grains when placed in the lower cup; determine the specific gravity of the substance, the weight of the hydrometer being 240 grains, and the weight required to sink it to the standard depth in the given fluid 30 grains.

The specific gravity required is ·51.

CHAPTER XV.

ON ATMOSPHERIC PRESSURE.

230. The atmosphere surrounding the earth, being a fluid acted on by gravity, presses upon all bodies immersed in it in accordance with the general laws of fluid pressure.

The pressure of the atmosphere upon any body will consequently vary with the depth of that body below its highest surface. As, however, the dimensions of any ordinary body upon the surface of the earth are inconsiderable, when compared with the height of the atmosphere, this pressure may, without any sensible error, be regarded as the same at all points in any such body

It is found, by experiment, that the average pressure which the atmosphere exerts upon a body, on or near the surface of the earth, is the same as that exerted by a column of water 32 feet in height, or by a column of mercury 30 inches in height; that is, it presses with a weight of about 15 lbs. upon every square inch of surface. That we are not, in ordinary circumstances, cognizant of this pressure arises from the characteristic property of fluids, in accordance with which they press in all directions upon bodies immersed in them; so that if the upper surface of a body, subject to the pressure of the atmosphere, experience a downward pressure of 15 lbs. to every square inch, the under surface experiences a corresponding upward pressure. If, however, we destroy this equilibrium, by removing or diminishing the pressure of the atmosphere upon one side of any body, we are immediately made sensible of the pressure which is exerted upon the opposite side. For instance, if a plate of glass be placed upon the top of a

cylindrical receiver connected with an air-pump, and the air be exhausted from the receiver, the plate of glass will be pressed down with a considerable force, and if not strong enough will be broken in pieces.

231. *When the atmosphere presses upon any point of a fluid mass, to determine the amount of transmitted pressure.*

If the atmosphere presses upon the highest surface of the fluid mass, then it is evident that the transmitted pressure is equal to the pressure of the atmosphere.

If the atmosphere presses at a point in the fluid mass below the highest surface, then a portion of the atmospheric pressure will be employed in counterbalancing the outward pressure of the fluid itself, and the remaining portion only of the atmospheric pressure will be transmitted by the fluid. Let h be the distance below the highest surface of the fluid mass of the point at which the atmosphere presses upon it, and let w be the weight of a cubic unit of the fluid; then, as seen in Chapter XIII., the fluid itself will exert at this point an outward pressure on every square unit $= wh$. Then if Æ denote the pressure exerted by the atmosphere upon a square unit, the transmitted pressure will be

$$Æ - wh$$

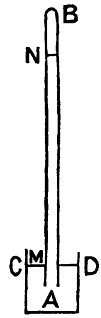
for every unit of surface of the fluid mass.

If the point at which the atmosphere presses upon the fluid mass be so taken that wh is greater than Æ, then it is plain that equilibrium cannot subsist, but motion will ensue amongst the particles of the fluid, until the distance becomes such that $wh = Æ$.

232. THE BAROMETER. This instrument is constructed in the following manner:—A glass tube, closed at one end, is filled with mercury, and, a finger being placed over the open end, is inverted in an open vessel of mercury.

Let AB be such a tube, let CD be the surface of the mercury in the vessel, and let MB be more than 31 inches. Then, since

CD is more than 31 inches below B, the highest surface of the fluid mass, the pressure of the fluid upon any square unit in the surface CD will be greater than the atmospheric pressure. The mercury will consequently fall in the tube. Let N be the point at which it rests, then if Æ be the atmospheric pressure, and w the weight of a cubic unit of mercury,



$$Æ = w \cdot MN.$$

The height of mercury in the tube will accordingly vary as the atmospheric pressure varies; will rise as the atmospheric pressure increases, and fall as it decreases.

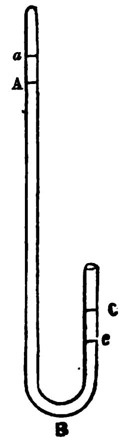
A graduated scale is attached to the tube AB, by means of which the changes in the height of the mercury in the tube can be observed and measured.

233. It will be seen, that when the mercury falls in the tube it rises in the cistern, and that consequently the fall of the mercury in the tube, as measured by the scale attached to it, will not correctly measure the variation in the barometrical column, but will be too great by the distance through which the mercury has *risen* in the cistern. In like manner, when the mercury rises in the tube, the variation, as measured by the scale, will be too small by the distance through which the mercury has *fallen* in the cistern.

If the cistern be large in comparison with the bore of the tube, the variation in the height of the mercury in the cistern is so small that it may be disregarded for ordinary purposes. When greater accuracy is desired, the bottom of the cistern is so made that it can be raised or lowered by a screw, and thereby the surface of the mercury in the cistern be adjusted to one uniform height.

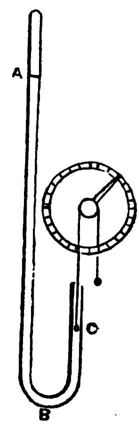
234. THE SIPHON BAROMETER. In this instrument the necessity

of any adjustment is avoided by the simple device of substituting for the cistern a small tube, of the same bore as that of the larger tube. Let ABC be such an instrument, the surfaces of the mercury being at A and C. Suppose now the mercury to fall in the shorter tube from C to c, and in consequence to rise in the longer tube from A to a. Then, since the bore of both tubes is alike, the distance Aa is equal to Cc. The entire variation in the barometric column is Aa + Cc, and therefore equal to 2 Aa. Hence the variation, as measured by the scale, is exactly one-half of the real variation.



This form of the instrument possesses some advantages, but is open to the objection that by diminishing the scale it increases the difficulty of observing small variations of the height of the column.

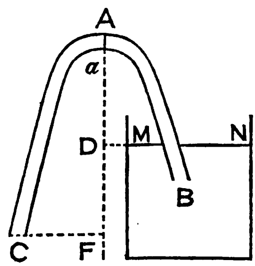
235. THE WHEEL BAROMETER. This form of the barometer has been contrived for the purpose of making small variations of the column more sensible. It consists of a siphon barometer, having a small bar of iron or glass floating upon the open surface of the mercury. A fine thread, attached to the ball, passes round a small wheel carrying an index, which moves over a graduated dial-plate, and the rise or fall of the mercury is measured by the arc of the circle over which the index moves. The variations of the barometric column range between 28 and 31 inches, or through a distance of three inches. As shown in the last section, the variation at C is one-half of the variation in the barometric column; and, consequently, the ball will rise or fall through a space of 1½ inch. If, then, the circumference of the wheel round which the ring passes be exactly 1½ inch, the index will make a complete revolution, when the ball



rises or falls through a height of $1\frac{1}{2}$ inch; and if the circumference of the dial-plate be divided into 360 equal parts, the motion of the index over 120 of these parts will correspond to a rise or fall of the ball through half an inch, and therefore to a change of one inch in the barometric column. Consequently, the variation of one 120th part of an inch is shown by the motion of the index over one division of the plate.

236. THE SIPHON. The siphon is a bent tube, open at both ends, and having one leg shorter than the other. If such a tube be filled with any fluid, and placed with the extremity of its shorter leg in a vessel containing the same fluid, the fluid will flow out through the longer leg.

Let ABC be the siphon filled with fluid, and having B, the extremity of its shorter leg AB, placed below MN, the surface of the fluid in the vessel. Let A be the highest point of the siphon, and through A draw the vertical line AF. Let MN produced meet this vertical line in D; and let CF be the horizontal line drawn through C.



Let Aa be a thin vertical film of the fluid passing through A. Since each side of this film is similarly situated with respect to A, the highest point of the fluid mass, the pressure arising from the weight of the fluid will be the same on each side. These pressures, therefore, are in equilibrium, and do not cause the motion of the fluid.

But if Æ be the atmospheric pressure upon a square unit, and w be the weight of a cubic unit of the fluid, then, since the atmosphere presses upon the surface MN at a distance AD below the highest point of the fluid mass, there is transmitted up the leg BA, by Art. 231, a pressure upon every square unit equal to Æ $- w$. AD.

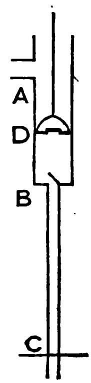
Similarly, since the atmosphere presses upon the surface of the

fluid mass at C, there is transmitted up CA a pressure upon every square unit equal to $Æ - w \,.\, AF$.

Hence, the one side of the film Aa will experience a pressure upon every square unit equal to $Æ - w \,.\, AD$, and the other side a pressure equal to $Æ - w \,.\, AF$. But since AD is less than AF, $Æ - w \,.\, AD$ is greater than $Æ - w \,.\, AF$; the film of fluid will therefore be forced down the leg AC. The same will happen to every film that may in succession occupy the position Aa; the fluid consequently will flow out at C, so long as there remains in the vessel any fluid above B.

It has been assumed in the preceding, that the height AD is such that $w \,.\, AD$ is less than Æ. If AD be so great that $w \,.\, AD$ is greater than Æ, then, as seen in the preceding Articles, the fluid will sink in the leg AB to some level below the bend, and the siphon consequently will cease to act.

237. The Common Pump. The common pump consists of a cylindrical barrel AB, furnished with a valve at B opening upwards, and with an air-tight piston, capable of being moved up and down by means of a rod.



The piston is also furnished with a valve opening upwards, and to the barrel at B is attached a pipe BC, called the suction pipe.

Let C be the surface of the water to be raised, and suppose the piston to be at B.

Let the piston be raised from B to A; the air in BC will, by virtue of its expansive force, press open the valve at B and fill the barrel AB; and in consequence of occupying a larger space, ABC will exert a less pressure than before upon the surface of the water within the pipe. Hence, the pressure of the air within the pump being less than the atmospheric pressure upon the surface of the water, the water will be forced up the pipe BC until equilibrium be restored.

Let the piston now descend; the valve at B will be closed, and

the air in AB will escape through the valve in D. If the piston be again raised, the preceding process will be repeated, and the water will again rise in the pipe BC. In like manner, upon every successive stroke of the piston the water will continue to rise in the pipe, until, if BC be less than 32 feet, it at last reaches B.

Let the water be at B, and let the piston be pressed down to B. Then, since the atmosphere presses upon the surface of the fluid at C, there will be transmitted by the fluid a pressure upon every square unit of the valve at B equal to $Æ - w \,.\, BC$ (Art. 231); and, consequently, when the piston is again raised, the water will force open the valve at B, and will rise in the barrel as far as A, if AC be less than 32 feet, or to some point between B and A, if AC be greater than 32 feet.

Upon the next descent of the piston, the valve in D will be forced open, and the water will flow through it; so that when the piston reaches B the water in the barrel will lie above the piston, and when the piston is raised will be raised along with it, and flow out at the spout A.

It will be seen, that if B be more than 32 feet above C, no water will rise into the barrel; and in such a case the pump will be useless. If B be less, but A more than 32 feet, the water will only in part fill the barrel, and the discharge of water during the ascent of the piston will not be continuous.

238. *In a common pump, whose spout is less than 32 feet from the surface of the water, to determine the resultant pressure upon the piston at any moment during its ascent, after it has commenced discharging the water.*

Let the piston be at D, then, since the water has commenced to be discharged, the barrel and pipe are filled with water, and since the piston is rising, the valve at B will be open, and that at D will be shut.

The pressure upon the upper side of the piston is the sum of the weight of the fluid mass in AD and the atmospheric pressure upon its surface. Therefore, upon every square unit of the upper

surface of the piston there is exerted a downward pressure equal to $Æ + w \cdot AD$, and upon every square unit of the lower surface of the piston there is, by Art. 231, an upward pressure equal to $Æ - w \cdot DC$. Let a be the number of square units in the area of the piston,

$\therefore$ the total pressure upon the upper surface $= aÆ + wa \cdot AD$,

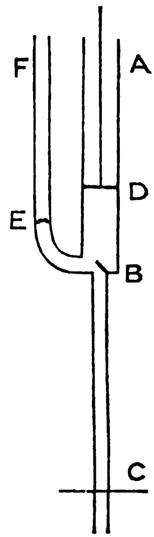
" " lower " $= aÆ - wa \cdot DC$.

The resultant pressure will be the difference of these pressures, and therefore equals

$$wa \cdot AC.$$

Therefore the resultant pressure is independent of the position of D; in other words, the pressure upon the piston is the same at every point in its ascent, after commencing to discharge, and is equal to the weight of a column of water, whose base equals the area of the piston, and whose height is the height of the spout of the pump above the surface of the water.

239. THE FORCING PUMP. The construction of the forcing pump is represented in the accompanying figure.



D is a piston with no valve, working in a barrel AB; BC is a pipe descending to the water; EF a tube leading upwards from B; at B and E are valves opening upwards.

If the piston be raised from B to A, then, as in the common pump, the air in BC will press open the valve at B and fill the barrel, and the water will rise to a certain height in BC. As the piston descends, the air in the barrel will be forced out through the valve at E; after a certain number of ascents the water will, if BC be less than 32 feet, pass into the barrel AB; and, upon its next descent, the piston will force the water in the barrel through the valve E into the pipe EF. As the piston continues to rise, more water will pass into the barrel, and

this in its turn will be forced, by the descent of the piston, into the pipe EF.

240. *To determine the resultant pressure upon the piston of a forcing pump, at any moment during its ascent or descent, after it has commenced discharging.*

First. Let the piston be ascending; the valve at B will be open, and that at E will be shut. Then, if Æ be the atmospheric pressure upon a square unit, w the weight of a cubic unit of water, and a the area of the piston, the downward pressure upon the upper surface of the piston will be aÆ, and the upward pressure upon the lower surface of the piston will be a(Æ $- w \cdot$ DC), or aÆ $- wa$DC. Therefore, the resultant pressure will act downwards, and be equal to

$$wa \cdot \mathrm{DC}.$$

Hence, since as the piston rises, DC increases, the resultant pressure increases at every moment of the ascent, and at any instant is equal to the weight of a column of water, whose base equals the area of the piston, and whose height is the distance of the piston from the surface of the water C.

Secondly. Let the piston be descending, the valve B will be closed, and that at E will be open. Let F be the highest point in the pipe EF.

The pressure upon the upper surface of the piston will, as before, be a downward force equal to aÆ; and the pressure upon the lower surface of the piston will be an upward force equal to aÆ $+ wa \cdot$ DF. The resultant pressure will therefore be an upward force equal to

$$wa \cdot \mathrm{DF}.$$

Hence, the pressure upon the piston increases at every moment of the descent, and at any instant equals the weight of a column of water, whose base equals the area of the piston, and whose height is the distance of the piston below the point at which the pipe discharges.

EXAMPLES.

1. The average height of the barometer at Paris, as obtained from a large number of observations, is 29·77 inches; determine the average pressure of the atmosphere.

The average pressure is 14 lbs. 10·3 oz.

2. A siphon is used for decanting a liquid, whose specific gravity is 1·7; determine the limit to the height of the highest point of the siphon above the surface of the fluid, when the barometer stands at 29 inches.

The required limit is 19 ft. 4 in.

3. The area of the piston of a common pump is 20 inches, and the height of the spout above the surface of the water is 24 feet; determine the pressure upon the piston.

The required pressure is 208 lbs. $5\frac{1}{3}$ oz.

4. If a pump be used to raise a fluid, whose specific gravity is 1·5; find the limiting height of the bottom of the barrel, when the barometer stands at 29·5 inches.

The limiting height is 22 feet $3\frac{1}{2}$ inches nearly.

5. If two vessels, containing fluids of different specific gravities which do not mix, be placed with their surfaces in the same horizontal plane; and if a siphon, having its legs filled respectively with the two fluids, be placed in the vessels, so that each leg is in the same fluid as that with which it is filled; show that the lighter fluid will force the heavier one entirely down the leg of the siphon, supposing that the level in either vessel is not perceptibly altered by the addition or removal of as much fluid as is contained in one leg of the siphon.

6. Two vessels, containing fluids of different specific gravities which do not mix, are placed so that the surfaces of the fluids are at different levels, and a siphon, having its legs filled respectively with the two fluids, is placed in the vessels with each leg in the same fluid as that with which it is filled; determine the position of the common surface of the two fluids, supposing, as before, that the level in either vessel is not appreciably altered by the addition or removal of as much fluid as is contained in one leg of the siphon.

Let s_1 and s_2 be the specific gravities of the fluids, and h_1 and h_2 the distances of the highest point of the siphon from the surfaces of the fluids; then the distance of the common surface below the highest point of the siphon is equal to

$$\frac{s_1 h_1 - s_2 h_2}{s_1 - s_2}.$$

7. Let the two vessels in example 5 be cylindrical and with equal bases, determine the position of the common surface of the two fluids when the alteration in the levels is not disregarded; also determine the amount of alteration in the levels, the siphon being of uniform bore.

Let s_1 and s_2 be the specific gravities of the two fluids, s_1 being that of the heavier. Let A be the area of the base of the vessels, and a that of a section of the siphon. Let h be the distance of the highest point of the siphon above the original level. Then the distance of the common surface below the highest point of the siphon is

$$\frac{(s_1 - s_2)\mathrm{A}h}{\mathrm{A}(s_1 - s_2) + a(s_1 + s_2)},$$

and the alteration of level is equal to

$$\frac{(s_1 - s_2)ah}{\mathrm{A}(s_1 - s_2) + a(s_1 + s_2)},$$

the lighter fluid being depressed, and the heavier fluid being raised by this amount.

CHAPTER XVI.

ON THE LAWS OF ELASTIC FLUIDS.

241. A gas or aëriform fluid is distinguished from other fluids by the tendency of its particles to recede from each other, that is, by an excess of the repulsive forces over the attractive forces acting upon its particles. If such a fluid be enclosed in any vessel, it will, like any other fluid, exert a pressure upon the sides of the vessel in consequence of the action of gravity upon its particles; but, irrespectively of this, it will exert another, and commonly far greater pressure, in consequence of the mutual repulsion of its particles. It is this latter pressure which is termed the *elastic force* of the gas. This is found to vary with the space within which the gas is enclosed, and also with the temperature. The law of this variation, as approximately determined by experiment, is exhibited in the following articles.

242. Law I. *The temperature remaining the same, the elastic force of any gas varies inversely as the space it occupies.*

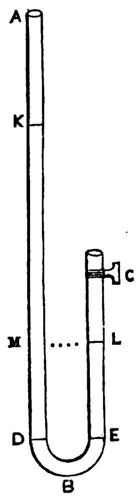
This law is often quoted as "Boyle and Marriotte's law," or simply as Boyle's law.

The experiments by which this law is established are of the following kind.

A glass tube of uniform bore, open at both ends, is bent into the form ABC; the shorter branch, BC, is furnished with a stop-cock C, and an accurately divided scale is attached to the longer branch.

Let the stop-cock be opened, and a small quantity of mercury

be poured into the tube. Let the surfaces of the mercury be at D and E; these will be in the same level, since the same pressure, viz., the atmospheric pressure, acts upon both. Let the stop-cock now be closed, the surfaces D and E will remain at the same level, and the pressure of the air in CE upon the surface at E is equal to the pressure of the atmosphere upon D; or the elastic force of the air in CE is equal to the atmospheric pressure. Let the barometer stand at h inches, and let w be the weight of a cubic inch of the mercury used in the barometer, then the atmospheric pressure per inch, and consequently the elastic force of the air in CE, is equal to



$$wh.$$

Now let mercury of the same temperature and density as that in the barometer be poured into the larger branch, and let the mercury rise to L in the shorter branch, and stand at K in the longer branch; then the pressure at L must be equal to the pressure at M. The pressure at M per inch is the atmospheric pressure increased by the weight of KM inches of mercury; hence, the elastic force of the air in CL is equal to

$$wh + w \,.\, \mathrm{KM}.$$

Hence we obtain this result,

$$\frac{\text{elastic force of the air in CL}}{\text{elastic force of the air in CE}} = \frac{wh + w\mathrm{KM}}{wh},$$

$$= \frac{h + \mathrm{KM}}{h}.$$

It is found by trial that

$$\frac{\mathrm{CE}}{\mathrm{CL}} = \frac{h + \mathrm{KM}}{h};$$

and therefore it follows that

$$\frac{\text{elastic force of air in CL}}{\text{elastic force of air in CE}} = \frac{\mathrm{CE}}{\mathrm{CL}},$$

or the elastic force of the air varies inversely as the space it occupies.

By similar experiments the law is established for other gases.

Hence, if p_1 be the elastic force, and V_1 the volume of a given mass of gas at a given temperature, and if p_2 be the elastic force when the same mass has become compressed or expanded into the volume V_2, the temperature remaining the same; then $p_1 : p_2 = \frac{1}{V_1} : \frac{1}{V_2}$

$$p_1 : p_2 :: V_2 : V_1,$$ and $= d_1 : d_2$

$$V_2 = \frac{V_1 p_1}{p_2}.$$

In performing the experiments described above, care must be taken to guard against any increase of temperature by the sudden compression of the gas. The mercury should be poured in very slowly, and some time allowed before observing the levels, that any heat which may have been generated may pass away by radiation.

243. Since the density of any body varies inversely as the space it occupies, it follows that Boyle and Marriotte's law may be thus enunciated. *The temperature remaining the same, the elastic force of any gas varies directly as the density.*

244. The following are examples in illustration of Boyle's law:—

Ex. 1. Determine the elastic force of a mass of air, whose volume is 100 cubic inches, when compressed into 48 cubic inches; the barometer standing at 30 inches.

The elastic force of the air before compression is equal to the atmospheric pressure, that is, to the weight of 30 inches of mercury, or 14·757 lbs. Hence, if p be the elastic force required,

$$p = \frac{100 \times 14{\cdot}757}{48},$$

$$= 30{\cdot}744 \text{ lbs.}$$

Ex. 2. An air-tight piston is fitted into a cylindrical vessel of uniform bore, closed at one end; the vessel is filled with air, and

the piston forced in with a pressure of 50 lbs. The length of the cylinder is 20 inches, the area of the piston 5 square inches, and the barometer stands at 29 inches; how far will the piston be forced in?

The elastic force of the air before compression is equal to the atmospheric pressure, that is, to the weight of 29 inches of mercury, or 14·265 lbs.

The elastic force of air after compression is equal to the atmospheric pressure increased by the applied pressure; the latter is $\frac{50}{5}$, or 10 lbs., hence the elastic force is 24·265.

Then if x be the distance through which the piston is forced, the reduced volume is to the original volume as $20-x : 20$; and therefore

$$\frac{20-x}{20}=\frac{14·265}{24·265};$$

whence,

$$x=\frac{200}{24·265},$$

$$= 8·243 \text{ inches.}$$

245. LAW II. *The pressure remaining the same, equal increments of temperature will cause in any gas equal increments of volume.*

This is called Dalton and Gay Lussac's law.

If V_0 be the volume of any gas at the freezing point, and aV_0 be the increment caused by an increase of one degree in the temperature, then $a\theta V_0$ is the increment caused by θ degrees; and hence, if V be the volume of the gas at the temperature of θ degrees above the freezing point,

$$V = V_0 + a\theta V_0$$

$$= V_0(1 + a\theta).$$

The quantity a is called the *co-efficient of expansion*, and for all gases its value is $\frac{1}{490}$ for the Fahrenheit scale, or $\frac{1}{273}$ for the centigrade scale.*

* In the Fahrenheit thermometer, the freezing point is marked 32°, and the degree is $\frac{1}{180}$ of the distance between the freezing and boiling points. In the centigrade thermometer, the freezing point is zero, and the degree is $\frac{1}{100}$ of the distance between the freezing and boiling points.

Hence, if the temperature of any quantity of gas be $\tau°$ Fahrenheit, then

$$\frac{\text{volume at } \tau°}{\text{volume at freezing point}} = 1 + \frac{\tau - 32}{490},$$
$$= \frac{458 + \tau}{490}.$$

Thus, for example, if 100 cubic inches of a gas at the freezing point be raised in temperature to 128° F., the pressure remaining the same, then

$$\text{the volume} = 100 \times \frac{458 + 128}{490},$$
$$= 119{\cdot}592 \text{ inches.}$$

Or if 100 inches of a gas at 80° F. be lowered in temperature to the freezing point,

$$\text{the volume} = 100 \times \frac{490}{458 + 80},$$
$$= 91{\cdot}078 \text{ inches.}$$

246. *To compare the volumes of a given quantity of gas at different temperatures, but under the same pressure.*

Let τ_1, τ_2 be the temperatures on Fahrenheit's scale, and let V_1, V_2 be the corresponding volumes of the gas. Then, by the preceding, if V_0 be the volume of the freezing point,

$$V_1 = V_0\{1 + a(\tau_1 - 32)\},$$

and $$V_2 = V_0\{1 + a(\tau_2 - 32)\};$$

therefore, $$\frac{V_1}{V_2} = \frac{1 + a(\tau_1 - 32)}{1 + a(\tau_2 - 32)}.$$

Substituting for a its value $\frac{1}{490}$, we have

$$\frac{V_1}{V_2} = \frac{458 + \tau_1}{458 + \tau_2}.$$

If the temperature be τ_1 and τ_2 on the centigrade scale,

$$\frac{V_1}{V_2} = \frac{1 + a\tau_1}{1 + a\tau_2},$$
$$= \frac{273 + \tau_1}{273 + \tau_2}.$$

247. *To compare the volumes of a given quantity of gas at different temperatures and under different pressures.*

Let V_1 be the volume of the gas, at a temperature θ_1 under a pressure p_1, and let V_2 be the volume of the same quantity of gas at a temperature θ_2, under a pressure p_2. Let the temperatures in both cases be reduced to the freezing point; then, by Art. 245, the volumes are respectively

$$\frac{V_1}{1+a\theta_1}, \text{ and } \frac{V_2}{1+a\theta_2}.$$

But since the temperatures are the same, the pressures are inversely as the volumes (Law i.), consequently,

$$p_1 : p_2 :: \frac{V_2}{1+a\theta_2} : \frac{V_1}{1+a\theta_1},$$

and hence,

$$\frac{V_1}{V_2} = \frac{p_2(1+a\theta_1)}{p_1(1+a\theta_2)},$$

or writing for a its value, and expressing the temperature by the Fahrenheit thermometer,

$$\frac{V_1}{V_2} = \frac{p_2}{p_1} \cdot \frac{458+\tau_1}{458+\tau_2}.$$

COR. Hence, if the volume remains unchanged when the temperature is altered,

$$\frac{p_1}{p_2} = \frac{458+\tau_1}{458+\tau_2}.$$

248. If ρ_1, ρ_2 denote the densities of the gas in the two cases, then, since the densities are inversely as the volumes, the result of the preceding section becomes

$$\frac{\rho_2}{\rho_1} = \frac{p_2(1+a\theta_1)}{p_1(1+a\theta_2)}$$

or,

$$\frac{p_1}{p_2} = \frac{\rho_1(1+a\theta_1)}{\rho_2(1+a\theta_2)},$$

which gives us the relation between the pressure, temperature, and density of any given quantity of gas.

The form of this result may be conveniently modified as follows:—Let ρ be the density at the freezing point under a standard pressure p. Then

$$\frac{p_1}{p} = \frac{\rho_1(1 + a\theta_1)}{\rho},$$

or,

$$p_1 = \frac{p}{\rho} \cdot \rho_1 (1 + a\theta_1).$$

But $\frac{p}{\rho}$ is a constant whose value may be determined by experiment for each particular gas. Let k denote the constant, and

$$p_1 = k\rho_1 (1 + a\theta_1);$$

or, as it may be written,

$$p = k\rho (1 + a\theta).$$

249. The following are examples in illustration of Dalton and Gay Lussac's law.

Ex. 1. Required the increase in the volume of a cubic foot of air, when the temperature is raised from 52° F. to 72° F., the pressure remaining unchanged.

Let V denote the increased volume; then, by Art. 246,

$$V = \frac{458 + 72}{458 + 52}$$

$$= \frac{530}{510}$$

$$= 1\tfrac{2}{51} \text{ cubic feet.}$$

Therefore the required increase is

$$\tfrac{2}{51} \text{ cubic feet,}$$

$$= 67{\cdot}76 \text{ cubic inches.}$$

Ex. 2. If a certain quantity of air have a volume of 100 cubic inches when the barometer stands at 30 inches, and the temperature is 52° F.; determine its volume when the barometer is at 29 inches, and the temperature 72° F.

Let V be the volume required, then, by Art. 247,

$$\frac{V}{100} = \frac{30}{29} \cdot \frac{458 + 72}{458 + 52}$$

$$= \frac{30}{29} \cdot \frac{530}{510};$$

$$\therefore \quad V = 107 \cdot 5 \text{ cubic inches.}$$

Ex. 3. If 100 cubic inches of air weigh 31 grains when the barometer is at 30 inches and the temperature 60° F.; determine the volume of 2 oz. of air when the barometer is at 29 inches and the temperature is at 122° F.

Since 100 cubic inches weigh 31 grains, $\frac{100 \times 875}{31}$ inches will weigh 875 grains, or 2 oz. Then, if V be the required volume, by Art. 247,

$$\frac{31V}{87500} = \frac{30}{29} \cdot \frac{458 + 122}{458 + 60};$$

$$\therefore \quad V = 3269 \cdot 4 \text{ cubic inches,}$$

$$= 1 \cdot 892 \text{ cubic feet.}$$

250. Law III. *When two gases, which do not chemically unite, are mixed together, each gas acts as a vacuum with respect to the other.*

If two vessels, containing two different gases which do not chemically unite, be brought together, and a communication be opened between them, each gas is found, after a short interval, to be equally diffused through both vessels. This result is independent of the specific gravities of the two gases, so that if the heavier gas be in the lower vessel, and the lighter one in the higher, the heavy gas will rise into the higher vessel, and the lighter gas will descend into the lower vessel.

In stating that each gas acts as a vacuum with regard to the other, Dr. Dalton, who first enunciated this law, did not mean to imply that the diffusion of one gas through the space occupied by another takes place with the same rapidity as if there had been an actual vacuum, but that the final result is the same.

251. *Given the volumes and elasticities of two gases to find the elastic force of the mixture.*

Let V_1 be the volume, and p_1 the elastic force of the gas A; and V_2 the volume, and p_2 the elastic force of the gas B.

When intermixture takes place, the gas A acquires the volume $V_1 + V_2$; and hence, by Law i.,

$$\text{the elastic force of A} = \frac{V_1 p_1}{V_1 + V_2}.$$

In like manner,

$$\text{the elastic force of B} = \frac{V_2 p_2}{V_1 + V_2};$$

but the total pressure on any unit of the containing surface is the sum of the pressures exerted severally by the two gases. Hence,

$$\text{elastic force of the mixture} = \frac{V_1 p_1 + V_2 p_2}{V_1 + V_1}.$$

Cor. i. If the two gases before mixture have the same elastic force, that is, if $p_2 = p_1$, then

$$\text{elastic force of the mixture} = \frac{V_1 p_1 + V_2 p_1}{V_1 + V_2} = p_1,$$

or the elastic force is unaffected by the mixture.

Cor. ii. If the volumes of the two gases be equal, or $V_2 = V_1$, then

$$\text{elastic force of the mixture} = \tfrac{1}{2}(p_1 + p_2).$$

EXAMPLES.

1. An air-tight piston is fitted into a cylindrical vessel filled with air; the length of the cylinder is 30 inches, and the area of the piston is 10 inches: required the force necessary to push the piston in through 15 inches, the barometer standing at 30 inches.

The force required is 147·57 lbs.

2. In the preceding, determine the force required to draw the piston out through 15 inches.

The force required is 49·19 lbs.

3. If the pressure of P lbs. will force a piston, whose area is a inches, into a cylindrical vessel containing air through a distance of c inches when the barometer stands at h_1 inches, how far will the same pressure force in the piston when the barometer stands at h_2 inches?

Let $\frac{P}{a}$ be equal to the weight of h inches of mercury, then the required distance is equal to

$$c \cdot \frac{h + h_1}{h + h_2}.$$

4. If 20 cubic inches of gas at 58° F. be heated to 110° F., determine the increase of volume.

The required increase is 2·015 inches.

5. If 142 cubic inches of gas at 110° F. cool down to 54°, determine the loss of volume.

The required loss is 14 inches.

6. A strong vessel is filled with air at a temperature of 52°, and heated to 212°; what is the elastic force of the enclosed air, the barometer standing at 30°?

The elastic force is equal to 19·479 lbs.

7. Assuming that 100 cubic inches of air weigh 31 grains when the barometer stands at 30 inches and the thermometer at 60° F., determine the weight of a cubic foot of air when the barometer is at 30·5 inches, and the thermometer at the freezing point.

The required weight is 575·7284 grains.

8. A cylindrical vessel, closed at the top, is sunk in water to a

given depth, determine the height to which the water rises in the vessel.

Let a be the height of the vessel, and h_1 the depth of the mouth of the vessel below the surface of the water. Let the atmospheric pressure be equal to the weight of a column of water whose height is h. Then, if w be the weight of a cubic unit of water, the atmospheric pressure is equal to

$$wh.$$

Let x be the height to which the water rises in the vessel, then the pressure at the surface of the water in the vessel, that is, at a depth of $h_1 - x$, is equal to

$$wh + w(h_1 - x),$$

which gives us the elastic force of the compressed air. But, by Law i., the elastic force is inversely as the space; therefore

$$\frac{wh + w(h_1 - x)}{wh} = \frac{a}{a - x},$$

or,

$$\frac{h + h_1 - x}{h} = \frac{a}{a - x}.$$

Whence, by solving this quadratic, we obtain

$$x = \tfrac{1}{2}(a + h + h_1) - \tfrac{1}{2}\sqrt{\{(a + h + h_1)^2 - 4ah_1\}}.$$

9. A diving bell, of cylindrical form, is sunk in water to a given depth, how much must the temperature of the enclosed air be raised that the water may not rise in the diving bell?

Let the temperature of the air when first enclosed be τ°. F. Let h_1 be the depth of the bottom of the diving bell, and h the height of a column of water equal to the atmospheric pressure; then the required increase of temperature will be equal to

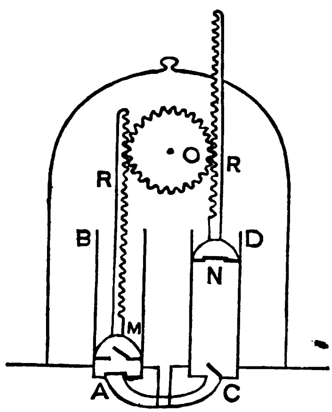
$$\frac{h_1(448 + \tau)}{h}.$$

CHAPTER XVII.

ON THE AIR-PUMP AND STEAM-ENGINE.

352. THE DOUBLE-BARREL AIR-PUMP. AB and CD are two cylindrical barrels connected together by the pipe AC, and communicating by means of another pipe with an air-tight receiver R. M and N are pistons furnished with valves opening upwards, and are worked by the toothed wheel O, so that when M descends N will ascend, and *vice versâ*. At A and C are valves opening upwards.

Let M be at A, and N at D, and the wheel be turned so that N may descend and M ascend. As N descends, the valve at C will close, and the air in the barrel CD will pass out through the valve in N. As M ascends, the pressure of the external air will close the valve in M; and the pressure being removed from the upper side of the valve A, the air in the receiver will, in consequence of its expansive force, press open the valve and fill the barrel AB. The air originally in the receiver, now occupying a larger space, viz. the receiver and barrel together, will be of diminished density.



If the wheel be again turned, a similar process will take place, and the air in the receiver will again expand, so as to fill both

receiver and barrel, and a second diminution of density will consequently take place.

The process may be continued so long as the air in the receiver can by its expansive force press open the valves at A or C; but since the expansive force of the air is diminished with the diminution of its density, the action of the pump must, after a certain number of strokes, entirely cease.

253. *To find the density of the air in the receiver after any number of turns of the wheel.*

If a quantity of air expand so as to occupy a space twice as great as before, its density will evidently be one-half its original density; if the space be three times as great, the density will be one-third the original density; and so, generally, the density after expansion will be to the density before expansion as the space occupied before expansion is to the space occupied after expansion.

Hence, if A be the content of the receiver, and B of each of the barrels, since at every turn the air in the receiver expands, and fills both receiver and barrel,

$$\frac{\text{density of the air at the close of any turn}}{\text{density of the air at the commencement of the turn}} = \frac{A}{A+B}.$$

Consequently,

$$\frac{\text{density of the air after 1 turn}}{\text{density of atmospheric air}} = \frac{A}{A+B}.$$

Also,
$$\frac{\text{density of the air after 2 turns}}{\text{density of the air after 1 turn}} = \frac{A}{A+B},$$

and $\therefore$
$$\frac{\text{density of the air after 2 turns}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^2.$$

In like manner,

$$\frac{\text{density of the air after 3 turns}}{\text{density of the air after 2 turns}} = \frac{A}{A+B},$$

and $\therefore$
$$\frac{\text{density of the air after 3 turns}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^3.$$

Hence, generally,

$$\frac{\text{density of the air after } n \text{ turns}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^n.$$

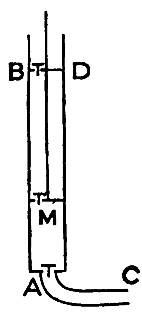
This result shows that, even if there were no practical limits to the working of an air-pump, the air could never be entirely exhausted; for the density of the air in the receiver always has some assignable ratio to the density of atmospheric air.

254. THE SINGLE-BARREL AIR-PUMP. This air-pump consists of a single barrel AB, communicating with an air-tight receiver by the pipe C.

The piston passes through an air-tight collar in the plate BD. At B is a valve opening upwards.

The process of exhaustion is precisely similar to that described in Art. 252.

The effect of the valve at B is to relieve the piston from the pressure of the atmosphere during a part of its ascent. For if the air in BM be of less density than atmospheric air, the pressure of the atmosphere will press down the valve at B, and will keep it closed until the piston by its ascent has so compressed the air in BM that its density has become greater than that of atmospheric air.

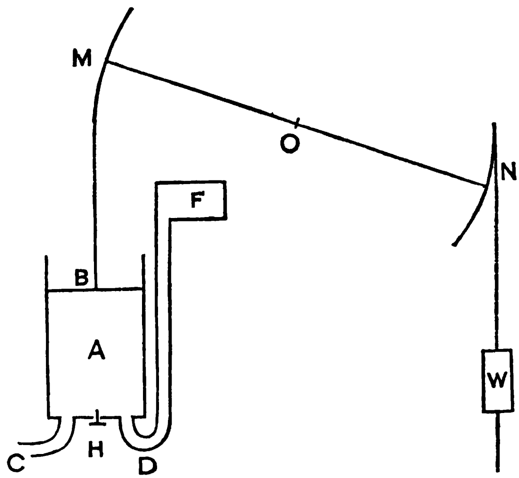


The plate BD, with its valve opening upwards, is not peculiar to the single-barrel air-pump, but may, if required, be adopted in a double-barrel air-pump. In fact, it is used in the more carefully-constructed double-barrel pumps.

255. THE ATMOSPHERIC STEAM-ENGINE. B is a solid piston working in the hollow cylinder A, and connected with one extremity of the beam MN by a chain attached to the piston-rod. C is a pipe leading from the boiler, and D a pipe leading from a reservoir of cold water F; C and D are both furnished with stopcocks. H is a valve opening downwards.

The beam MN moves about its centre O, and to the extremity N is attached a weight or counterpoise W, equal to half the atmospheric pressure upon the surface of the piston B.

Suppose the piston B to be at the bottom of the cylinder, and let the pressure exerted by the steam in the boiler be equal to that exerted by the atmosphere. If the cock in C be now opened, the steam will enter the cylinder, and will press upon the under



surface of the piston with a force equal to the atmospheric pressure upon the upper surface. The weight W will consequently descend, and by its descent will raise the piston B to the top of the cylinder.

Let the cock in C be now closed, and that in D be opened; a jet of cold water will issue into the cylinder, and condensing the steam in A will leave a vacuum below B. Then, since the pressure of the atmosphere upon B is double the weight W, B will be pressed down with a force equal to W.

When B arrives at the bottom of the cylinder, the cock in D is closed, and that in C is opened, and the piston ascends as before.

The water in the cylinder escapes by the valve H, which is forced open by the pressure of the steam when first admitted, and is conveyed by a pipe into a cistern communicating with the boiler.

256. THE SINGLE-ACTING STEAM-ENGINE. This engine differs from that described in the preceding article in employing, instead of atmospheric pressure, the pressure of steam to impel the piston downwards. It is thus in the strictest sense a *steam*-engine.

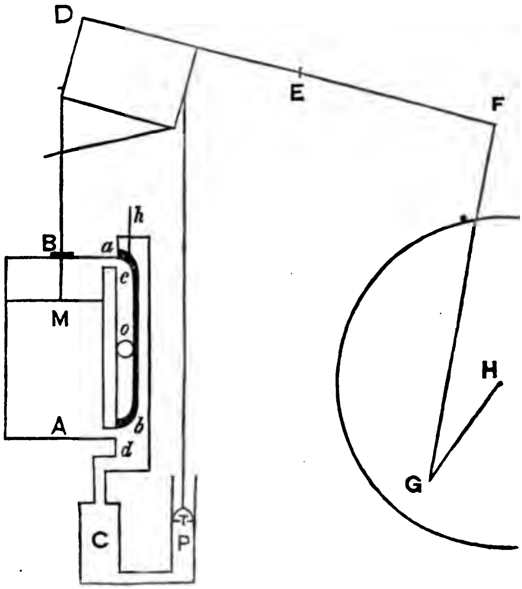
The top of the cylinder is closed, with the exception of a passage for the piston-rod. This passage is furnished with a stuffing-box filled with tow saturated with grease, so that while the piston-rod can move freely up and down, the steam is prevented from escaping. The steam is admitted into the cylinder above the piston, at the moment that a vacuum has been produced below the piston by condensation; the pressure of the steam then forces the piston downwards. When the piston has arrived at the lowest point, the steam which fills the cylinder above the piston is allowed to circulate below the piston (a commuication between the upper and lower parts of the cylinder being opened for this purpose). The piston is then pressed equally by the steam upon its upper and lower surfaces, and will consequently be raised by the weight attached to the beam.

In the single-acting steam-engine the pressure of the steam—as in the atmospheric engine the pressure of the atmosphere—acts only during the descent of the piston; and in both engines the force necessary to raise the counterpoise was so much subtrated from the available power of the engine. These objections were obviated by the invention of the double-acting steam-engine, in which the upward as well as the downward stroke was caused by the pressure of the steam.

257. THE DOUBLE-ACTING STEAM-ENGINE. AB is a hollow cylinder closed at both ends. M is a solid piston. The piston-rod passes through a stuffing-box at B, and is connected with one extremity of the beam DEF; the other extremity of the beam is

attached to the crank of a fly wheel. C is a vessel, called the condenser, surrounded with cold water, *ad* is a box communicating with the upper and lower parts of the cylinder AB, with the condenser C, and also, by means of a steam-pipe passing from an orifice at *o*, with the boiler.

A moveable slide *ab* works in the box *ad*, and regulates the passage of the steam from the boiler into the cylinder, and from the cylinder into the condenser. When the slide is in the position



represented in the figure, the steam from the boiler can pass into the upper part only of the cylinder, and the steam below the piston can escape into the condenser. If the slide be pressed down by means of the rod *h*, so that *a* shall come to *c* and *b* to *d*, the steam from the boiler can pass into the lower part only of the cylinder, and the steam above the piston can escape into the condenser.

Suppose M to be at the top of the cylinder, and the space below M to be filled with steam from the boiler. Let the slide be moved into the position represented in the figure. The steam below the piston will pass into C, and being there condensed will leave a vacuum in AM. At the same time the steam from the boiler will pass into the cylinder above the piston, and the pressure of the steam will force the piston down.

Let the slide be now moved into its lower position. The steam above the piston will flow into the condenser, and leave a vacuum in BM. At the same time the steam from the boiler will enter the cylinder below the piston, and the piston will be again raised.

The water formed in C by the condensation of the steam is removed by a pump P.

258. The engines described in the preceding articles may be arranged under the general class of *condensing steam-engines.* In engines of this class, the condensation of steam is employed to destroy or to diminish the pressure upon one side of the piston. Other engines are termed *non-condensing.*

259. THE NON-CONDENSING STEAM-ENGINE. In this engine the condenser and the somewhat cumbrous apparatus connected therewith are not required. The structure of the engine is in consequence considerably simplified, and its weight greatly diminished. And for these reasons locomotive engines are always non-condensing.

The non-condensing steam-engine is worked by steam of a pressure greater than the atmospheric pressure. Let steam of such a pressure enter the upper part of the cylinder, and at the same time let a communication be opened between the lower part of the cylinder and the atmosphere. The pressure of the steam on the upper surface of the piston being greater than the atmospheric pressure on the lower, the piston will descend.

Let the steam from the boiler now enter the lower part of the cylinder, and let a communication be opened between the upper part of the cylinder and the atmosphere. The steam in the upper

part of the cylinder will escape into the atmosphere, and the pressure upon the lower surface of the piston being greater than that upon its upper surface, the piston will ascend.

Non-condensing engines are popularly, but improperly, called *high-pressure steam-engines*, and, in contradistinction, the condensing engines are called *low-pressure steam-engines*.

MISCELLANEOUS EXAMPLES.

1. The arms of a lever of the first kind, of uniform thickness and density, are a and b, the weight of a unit of the bar is m, what weight must be attached to the extremity of the shorter arm that it may balance the longer arm?

The weight required is equal to $\frac{m}{2} \cdot \frac{a^2 - b^2}{b}$.

2. If the inclination of a plane be 30°, show that the time of descent is equal to that of a body falling freely through twice the length of the plane.

3. If a solid body, whose specific gravity is s, be placed in a vessel containing two fluids that do not mix, whose specific gravities are respectively s_1 and s_2, s_1 being greater and s_2 being less than s, show that the part immersed in the heavier liquid is to the whole solid as $s - s_2 : s_1 - s_2$.

4. If a triangle, whose height is h, be divided by a line drawn parallel to its base, at a distance from the vertex of one nth of the total height, what is the distance of the centre of gravity of the lower portion from the base of the triangle?

The distance required is equal to $\frac{h}{3}\left(1 - \frac{2}{n(n+1)}\right)$.

5. If a pyramid, whose height is h, be divided by a plane drawn parallel to its base at a distance from the vertex of one nth of the total height, what is the distance of the centre of gravity of the frustum from the base of the pyramid?

The distance required is equal to $\frac{h}{4}\left(1 - \frac{3}{n(n^2+n+1)}\right)$.

6. If a cone be immersed in any fluid so as to be just covered, show that the total pressure upon the conical surface, when the vertex is upwards, is twice as great as when the vertex is downwards.

7. Two cords, bearing equal weights P, P, and passing over two pulleys A, B, which lie in the same horizontal line, at a distance from each other equal to $2a$, are attached at a point C to a weight W, what in the position of equilibrium is the distance of C below the horizontal line through A and B?

The distance required is equal to $\frac{Wa}{\sqrt{(4P^2 - W^2)}}$.

8. If the side of a vessel filled with fluid be a rectangle, and it be divided by a horizontal line drawn at one nth of the depth; show that the pressure on the lower portion is to that on the upper as $n^2 - 1 : 1$.

9. Two fluids that do not mix, and whose specific gravities are s_1, s_2, are poured into a small bent tube of uniform bore, in such quantities as would fill respectively a and b inches of the tube; what is the distance of the common surface of the fluids from the bottom of the tube?

The distance required is $\frac{as_1 - bs_2}{2s_1}$ if as_1 be greater than bs_2, but $\frac{bs_2 - as_1}{2s_2}$ if bs_2 be greater than as_1.

10. If a, b, c be the sides of a triangle, and k, l, m the lines

drawn from the centre of gravity to the vertices; show that $a^2 + b^2 + c^2 = 3\,(k^2 + l^2 + m^2)$.

11. If a, b, c, d, e, f be the edges of a triangular pyramid, and k, l, m, n the distances of the centre of gravity from each vertex; show that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 4\,(k^2 + l^2 + m^2 + n^2).*$$

✓ 12. If a triangular plate float vertically in any fluid, show that, in the position of equilibrium, the lines joining the bisection of that side which is either wholly within or wholly without the fluid, with the points at which the other sides meet the surface of the fluid, will be equal to one another. Proved in Art. 40.

13. The side of a vessel filled with any fluid is a triangle, having its vertex at the bottom of the vessel, and is divided by a horizontal line drawn at one nth of the depth; show that the pressure on the whole side is to that on the lower portion as

$$n^3 : (n-1)^2\,(n+2).$$

14. A common pump is worked by a lever, whose arms are p and q, the spout is h inches above the surface of the water; what must be the area of the piston that the pump may be worked by a power P?

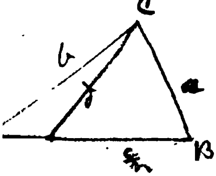
The area of the piston is $\dfrac{Pp}{qh} \cdot \dfrac{1728}{1000}$ inches.

15. A body is projected with a velocity V up a smooth inclined plane, rising h in l; determine the length of the ascent.

The required distance is equal to $\dfrac{V^2 l}{2gh}$.

* Before attempting this example or the preceding, prove the following theorem by aid of the Second Book of Euclid. If in any triangle ABC a point D be taken in the base AB, such that DB is one nth part of AB, then

$$CD^2 = \frac{n \cdot AC^2 + n\,(n-1) \cdot BC^2 - (n-1)\,AB^2}{n^2}.$$



$$d^2 = a^2 + \frac{c^2}{n^2} - \frac{2ac}{n}\cos B = a^2 + \frac{c^2}{n^2} - \frac{2ac}{n} \cdot \frac{b^2 - a^2 - c^2}{2ac}$$

$$= \frac{n b^2 + n(n-1)a^2 - (n-1)c^2}{n^2}$$

16. An inelastic body falling on to an inclined plane from any point in the perpendicular to the plane drawn through its highest point, will reach the bottom of the plane in the same time, and with the same velocity, as a body falling down the whole length of the plane.

17. A beam of uniform thickness, whose length is l, and weight W, is placed with one end upon a horizontal plane, and the other upon the summit of an inclined plane, whose height is h and base b; show that the horizontal force to be applied at the foot of the beam to keep it at rest is

$$\frac{W}{2} \cdot \frac{h\sqrt{(l^2-h^2)}}{h^2+b\sqrt{(l^2-h^2)}}.$$

18. A triangular plate ABC, having the angle ACB a right angle, is fastened by a pivot at C, and has weights P and Q suspended from A and B; show that, in the position of equilibrium, if $AC=a$, and $BC=b$, and if x and y be the distances of A and B severally below the horizontal line drawn through C,

$$x=\frac{a^2P}{\sqrt{(a^2P^2+b^2Q^2)}} \text{ and } y=\frac{b^2Q}{\sqrt{(a^2P^2+b^2Q^2)}}.$$

19. If, in the preceding, ABC be an equilateral triangle, whose side is a, prove that in the position of equilibrium,

$$x=\frac{a}{2} \cdot \frac{2P+Q}{\sqrt{(P^2+Q^2+PQ)}} \text{ and } y=\frac{a}{2} \cdot \frac{2Q+P}{\sqrt{(P^2+Q^2+PQ)}}.$$

20. If a straight line be drawn from the highest point of a given circle to a given point within it, and be produced to meet the circle, the produced part is the line of quickest descent from the given point to the circle.

21. If h_1, h_2, h_3 be the distances of the vertices of a triangle from any plane, show that the distance of the centre of gravity, from the same plane, is equal to

$$\tfrac{1}{3}(h_1+h_2+h_3).$$

22. The two parallel sides of a trapezium are at right angles with the base; required the distance of the centre of gravity from the base.

Let h_1 and h_2 be the lengths of the two parallel sides, then the distance required will be equal to

$$\frac{1}{3} \cdot \frac{h_1^2 + h_1 h_2 + h_2^2}{h_1 + h_2}.$$

23. If equal weights of three substances, whose specific gravities are s_1, s_2, and s_3 be compounded, without any change of volume, and if s be the specific gravity of the compound, show that

$$\frac{1}{s} = \frac{1}{3}\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3}\right).$$

24. A body composed of n different substances, whose volumes are V_1, V_2, and specific gravities s_1, s_2, &c., just floats in water; show that

$$V_1 (1 - s_1) + V_2 (1 - s_2) + \&c. \ldots\ldots + V_n (1 - s_n) = 0.$$

25. A body whose weight is W, composed of two substances, A and B, displaces, when entirely immersed, V inches of water; corresponding weights of A and B displace V_1 and V_2 inches: determine the weight of A contained in the compound.

The weight required is equal to

$$W \cdot \frac{V - V_2}{V_1 - V_2}.$$

26. If 100 cubic inches of air weigh 31 grains, show that the weight of air contained in a globe one foot in diameter is very accurately one 25th part of a pound avoirdupois.

27. If the volume of A is m times the volume of B, and the specific gravity of A is n times the specific gravity of B, show that the weight of A is mn times the weight of B.

28. The content of the receiver of an air-pump is 5 times that of the barrel; determine the elastic force of the air after 10 strokes of the piston, the atmospheric pressure being estimated at 15 lbs. to the square inch.

The elastic force is 2·4226 lbs.

29. A barometer standing at 30 inches is placed under the receiver of an air-pump; at what height will the mercury stand after 12 strokes of the piston, the content of the receiver being 9 times that of the barrel?

The required height is 8·4728 inches.

30. What volume of water must be added to a pint of fluid whose specific gravity is ·52 that the specific gravity of the mixture may be ·68?

The required quantity of water is half-a-pint.

31. A body falls from the highest to the lowest point of a circle whose plane is vertical along two of the sides of the inscribed square; supposing that there were no loss of velocity in passing from the first side to the second, show that the time of descent would be twice that of falling freely down the radius of the circle.

32. A smooth hollow tube ABC, having the parts AB and BC equal, and bent so that the angle ABC is equal to 120°, is placed with its lower leg BC vertical; determine the time in which a ball just fitting the tube will fall from A to C.

Let a be the length of each of the legs of the tube, then the time required is equal to

$$3\sqrt{\left(\frac{a}{g}\right)}.$$

33. In the preceding, if the upper leg be vertical, show that the time of descent is

$$\sqrt{\left(\frac{6a}{g}\right)}.$$

34. In the year 1670, a scheme was proposed by Francis Lana for navigating the atmosphere by means of four large copper balls from which the air had been exhausted. Each ball was to be 25 feet in diameter, and the metal to weigh 365 lbs.; determine the ascending power.

The ascending power of each ball is 261 lbs.

35. A silk balloon, 40 feet in diameter, and weighing 80 lbs., is filled with hydrogen gas of specific gravity ·0694 (sp. gr. of air = 1); determine the weight required to hold the balloon down.

The weight required is 2306 lbs.

36. A substance weighs w_1 in a liquid whose specific gravity is s_1, and w_2 in a liquid whose specific gravity is s_2; determine the specific gravity and weight of the substance.

The required sp. gr. is equal to $\frac{w_1 s_1 - w_2 s_2}{w_1 - w_2}$, and the weight is equal to $\frac{w_1 s_1 - w_2 s_2}{s_2 - s_1}$.

Part III.

STATICS.

CHAPTER I.

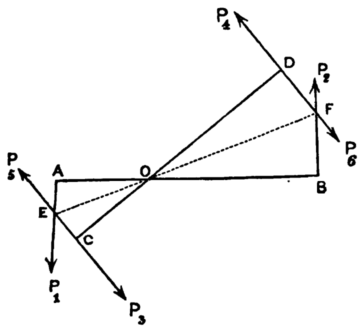
ON THE THEORY OF COUPLES.

260. Definitions. It has been seen that two equal non-concurrent parallel forces do not admit of a single resultant, but form what is termed a statical *couple*, having a tendency to produce a rotatory motion. The perpendicular distance between the directions of the two forces is termed the *arm* of the couple; the product of either force, and the arm, is termed the *moment* of the couple. Let the direction in which the hands of a watch move be taken as the positive direction of rotation, then if a couple tend to produce rotation in this direction, it is termed a *positive couple;* if in the contrary direction, a *negative couple.* The direction in which any rotatory motion appears to take place depends upon the position of the observer. Every couple is therefore positive or negative, according as it is viewed from the one side of its plane or from the other side. If it appear as a positive couple, that face of its plane upon which the observer is looking is termed its *positive face;* and if it appear as a negative couple, the face is termed the *negative face.*

A line perpendicular to the plane of a couple, in length representing the moment, is termed the *axis* of the couple. The axis is positive when pointing away from the positive face, and negative when pointing away from the negative face.

261. *A couple may be turned in its own plane through any angle about any point in its arm, without altering its statical effect.*

Let AB be the arm of the couple, P_1 and P_2 the forces. Let O be any point in AB, and let the arm revolve about O into any new position CD. At C introduce two equal and opposite forces P_3 and P_5 acting perpendicularly to CD, and each equal to P_1 or P_2. Similarly, at D introduce P_4 and P_6. The line EO bisects the angle AEC, and FO the angle BFD, also EO and FO are in the same straight line.



P_1 and P_5 are equivalent to some force in EO,

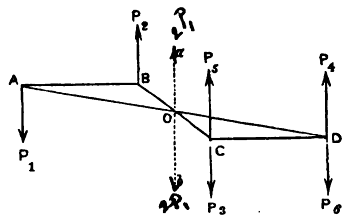
P_2 and P_6 " an equal force in FO;

therefore P_1, P_2, P_5, P_6 are in equilibrium, and may be removed. Removing them, there will remain the forces P_3, P_4 acting in the arm CD. Hence, the couple P_3, CD, P_4 produces the same statical effect as the couple P_1, AB, P_2.

262. *A couple may be moved into any position parallel to itself, either in the same or in a parallel plane, without altering its statical effect.*

Let P_1, AB, P_2 be the original couple. Let CD be any line equal and parallel to AB. At C introduce two equal and opposite forces P_3 and P_5, each parallel and equal to P_1. Similarly, at D, introduce P_4 and P_6.

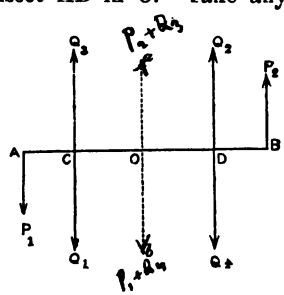
Join A, D and B, C. These lines will bisect each other at O. P_1 and P_6 are equivalent to $2P_1$ acting in the direction O*b*. P_2 and P_5 are equivalent to $2P_1$ acting in the direction O*a*. Hence, P_1, P_2, P_5, P_6 are in equilibrium, and may be removed. Removing them there remains only the couple P_3, CD, P_4, which consequently produces the same statical effect as the original couple.



263. By combining the two preceding sections, we see that a couple may be removed to any position in its own or a parallel plane, without altering its statical effect; for, by the former section, it may be turned round until its arm be parallel to the new position, and then, by the latter section, it may be directly removed into that position.

264. *Two couples, in the same or parallel planes, and tending to produce rotation in the same direction, are equivalent, if their moments are equal.*

Let P_1, AB, P_2, be any couple. Bisect AB in O. Take any points C, D, equidistant from O. At C and D introduce two pairs of equal and opposite forces, Q_1, Q_3, and Q_2, Q_4; and let Q_1, Q_2, Q_3, Q_4 be of such magnitude that each, when multiplied by CD, $= P_1 \times AB$.



Since $Q_3 \times CD = P_2 \times AB$, it follows that $Q_3 \times CO = P_2 \times BO$.

Therefore, P_2 and Q_3 are equivalent to $P_2 + Q_3$, acting in the direction O*a*.

Similarly, P_1 and Q_4 are equivalent to $P_1 + Q_4$, acting in the direction O*b*.

Therefore the forces P_1, P_2, Q_3, Q_4 are in equilibrium, and may be removed. Removing them, there will remain only the couple Q_1, CD, Q_2 tending to produce rotation in the same direction as the original couple, and (since $Q_1 \times CD = P_1 \times AB$) having an equal moment.

265. *To find the resultant of any number of couples acting in the same or parallel planes.*

By Art. 264, the couples may be all replaced by others having equal arms; and, by Art. 263, they may then be removed until their arms coincide. At each extremity of the common arm there will then be a corresponding set of forces acting in the same straight line, and whose resultant is consequently their algebraic sum.

Let P, Q, R, &c. be the forces, and a, b, c, &c. their arms. Resolve these into couples having a common arm m, and let P' Q', R', &c. be the reduced forces; then the resultant of all the couples will be a couple whose moment is

$$(P' + Q' + R' + \&c.)\, m;$$

but $P'm = Pa$ and $Q'm = Qb$, and so on. Therefore,

$$(P' + Q' + R' + \&c)\, m = Pa + Qb + Rc + \&c.;$$

whence the resultant is a couple whose moment equals the algebraic sum of the moments of the component couples.

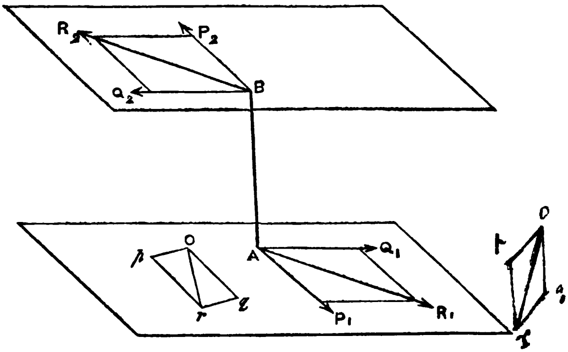
266. *To find the resultant of two couples not acting in the same or parallel planes.*

If the arms of the given couples be unequal, replace them by others having equal arms. Let the planes of the couples intersect in the line AB, and let this be the common arm of the couples, and let the reduced couples be P_1, AB, P_2, and Q_1, AB, Q_2. The planes of P_1, Q_1, and P_2, Q_2, are each perpendicular to AB, and are therefore parallel.

Let R_1 be the resultant of P_1 and Q_1, and R_2 the resultant of

P_2 and Q_2; then R_1 and R_2 are equal and parallel, and consequently the resultant of the given couples is a couple R_1, AB, R_2.

Through any point O in the plane of P_1, Q_1, draw the lines Op and Oq at right angles to the directions of P_1 and Q_1, and both



pointing away from the same face of the planes of the component couples. Make Op and Oq proportional respectively to P_1 and Q_1; then will the diagonal Or be perpendicular and proportional to R, and point away from the same face of the resultant couple. But since Op, Oq, and Or are proportional to P_1, Q_1, and R_1, they are also proportional to $P_1 \times AB$, $Q_1 \times AB$, and $R_1 \times AB$, that is, they are proportional to the moments of the several couples; therefore, Op, Oq, and Or are the axes of the three couples. Hence, *if two sides of a parallelogram meeting in any point represent the axes of two couples, both being positive or both negative axes, the diagonal drawn through the same point represents the corresponding axis of the resultant couple.*

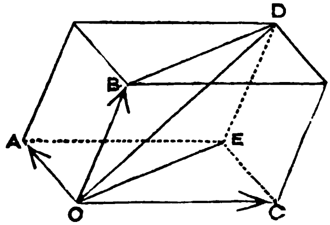
Let θ be the angle between P and Q, then the angle $pOq = \theta$. Hence, if L and M be the moments of the component couples, and G the moment of the resultant couple,

$$G^2 = L^2 + M^2 + 2\,LM \cos\theta.$$

267. *To find the magnitude and position of the resultant of three couples acting in planes at right angles to each other.*

Since the planes of the couples are at right angles to each other, their axes will also be at right angles to each other.

Let AO, BO, and CO represent the axes of the couples, then, by the preceding Article, EO will represent the axis of the couple which is the resultant of the couples whose axes are AO and CO; and, in like manner, DO will be the axis of the resultant of this and the third couple, and, therefore, of the three given couples.



Let L, M, N, be the moments of the given couples, and G the moment of the resultant couple, then

$$G^2 = L^2 + M^2 + N^2.$$

Let λ, μ, ν, represent the angles which the axis of the resultant couple makes with the axes of the component couples.

$$\cos \lambda = \frac{AO}{DO} = \frac{L}{G},$$

$$\cos \mu = \frac{BO}{DO} = \frac{M}{G},$$

$$\cos \nu = \frac{CO}{DO} = \frac{N}{G}.$$

268. *If any number of forces in equilibrium can be reduced into two sets of parallel forces, the one set being not parallel to the other, and the two sets not separately in equilibrium; then the resultant of each set is a couple, and the moments of the two couples are equal.*

For, if possible, let each set have a single resultant; then, since the resultant of any number of parallel forces is parallel to the components, we shall have two forces in equilibrium which, by the hypothesis, are inclined to each other. This is impossible, and therefore each set cannot have a single resultant.

Nor is it possible that one set should have a single force for its resultant, and the other a couple; for then a force and a couple would be in equilibrium.

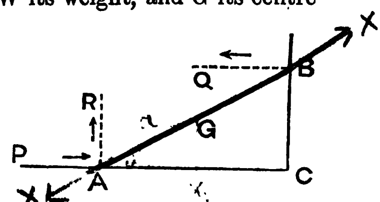
It follows, therefore, that the resultant of each set is a couple; and since there is equilibrium the moments of the two couples must be equal.

Cor. Hence, if four forces acting upon a rigid body be in equilibrium, and the first and second be parallel, and likewise the third and fourth, but the latter pair not parallel to the former; then is the first equal to and non-concurrent with the second, and the third equal to and non-concurrent with the fourth.

269. The theorem given in the last Article affords a ready means for the solution of many statical problems. The following is an example.

Ex. *One extremity* B *of a beam* AB *rests against a smooth vertical wall* BC, *and the other upon a smooth horizontal plane* AC; *required the horizontal force to be applied at the lower extremity, in order that the beam may rest in a given position; also the pressures on the wall and plane.*

Let l be the length of the beam, W its weight, and G its centre of gravity. Let AG $= a$, and the angle BAC $= \alpha$. Let P be the horizontal force required. Let R denote the resistance of the plane at A, and Q the resistance of the wall at B.



We have then four forces in equilibrium, forming two pairs of parallel forces, viz., R at A parallel to W at G, and P at A parallel to Q at B. Hence, by the preceding,

$$R = W,$$

$$Q = P;$$

and, since the moments of the two couples are equal,

$$Pl \sin \alpha = Wa \cos \alpha,$$

and $\therefore$

$$P = \frac{Wa \cos \alpha}{l}.$$

EXAMPLES.

1. If three forces act upon a rigid body along the sides of a triangle, and are also represented in magnitude and direction by the sides of the triangle taken in order, the resultant is a couple whose moment is equal to twice the area of the triangle.

2. If forces act upon a rigid body along the sides of any rectilinear figure, and are also represented in magnitude and direction by the sides of the figure taken in order, the resultant is a couple whose moment equals twice the area of the figure.

3. If four forces act upon a rigid body along the sides of the figure formed by joining the extremities of two intersecting lines, and be represented in magnitude and direction by the sides of the figure taken in order, show that the resultant is a couple whose moment is equal to twice the difference of the vertically opposed triangles.

4. A beam of uniform thickness and density, whose weight is W and length $2a$, rests with one extremity A upon a smooth horizontal plane, and in contact with two pegs, one at B above the beam, and another at C below it; find the pressures on the plane and pegs.

Let b be the distance between the pegs, and α the inclination of the beam to the plane; then the pressure on the plane is equal to

$$W,$$

and the pressure on each of the pegs is equal to

$$\frac{Wa \cos \alpha}{b}.$$

5. A beam leaning against a fixed peg, and with one extremity A upon a smooth horizontal plane, is kept at rest by a power P acting at A, at right angles with the beam; required the value of P when the beam is inclined to the plane at a given angle α.

Let h be the height of the ~~pivot~~ peg above the horizontal plane, W the weight of the beam, and a the distance of the centre of gravity of the beam from A; then the required ~~distance is~~ equal to

$$\frac{Wa \sin 2\alpha}{2h}.$$ force

6. The lower extremity of a weightless beam moves freely about a fixed pivot, a weight W is suspended from the other extremity, and the beam is supported in a given position by a cord attached at its centre at right angles to the beam; required the tension in the cord and the pressure upon the pivot.

Resolve the resistance of the pivot into two components acting parallel respectively to the tension and the weight, and apply the theorem of Art. 268.

Let α be the inclination of the beam to the horizontal line, then the tension in the cord is equal to

$$2W \cos \alpha,$$

and the pressure on the pivot is equal to

$$W.$$

7. A beam, of uniform thickness and density, rests with one extremity upon a smooth horizontal plane, and the other upon a smooth vertical plane; the inclination of the plane is 60°; determine the horizontal pressure which must be applied at the lower end of the beam that the inclination of the beam may be 30°, also the pressures on the planes.

Let W be the weight of the beam, then the horizontal force required is equal to

$$\frac{W\sqrt{3}}{4},$$

the pressure upon the inclined plane is equal to

$$\frac{W}{2},$$

and the pressure upon the horizontal plane is

$$\frac{3W}{4}.$$

Grind.

CHAPTER II.

ON THE GENERAL EQUATIONS OF EQUILIBRIUM.

270. *To find the magnitude and direction of the resultant of three forces acting upon a point in directions at right angles to each other.*

Let AO, BO, CO (fig. Art. 267) represent the forces; then, if X, Y, Z be their several magnitudes, and R that of their resultant,

$$R^2 = X^2 + Y^2 + Z^2;$$

and if a, b, c be the angles which the direction of the resultant makes with the directions of the components,

$$\cos a = \frac{X}{R}, \ \cos b = \frac{Y}{R}, \ \cos c = \frac{Z}{R}.$$

271. The equations just obtained give $X = R \cos a$, $Y = R \cos b$, $Z = R \cos c$; and hence, if any force R make angles a, b, c with three rectangular axes, it may be resolved into forces $R \cos a$, $R \cos b$, $R \cos c$, acting along the three axes respectively.

272. *To find the magnitude and direction of the resultant of any number of forces acting upon a point.*

Let P_1, P_2, &c. be the forces, and α_1, β_1, γ_1, α_2, β_2, γ_2, &c. be the angles which they severally make with three rectangular axes, which for distinction call the axes of x, y, and z respectively. Let P_1 be resolved into forces acting along the axes; by the preceding Article, these will be $P_1 \cos \alpha_1$, $P_1 \cos \beta_1$, $P_1 \cos \gamma_1$. Proceeding in like manner with the other forces, and remembering

that the resultant of any number of forces acting along the same line is their algebraic sum, we obtain

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. \text{ acting along axis of } x,$$
$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \&c. \quad ,, \quad ,, \quad y,$$
$$P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \&c. \quad ,, \quad ,, \quad z.$$

The sum of a series of terms of the same form is conveniently expressed by the symbol Σ prefixed to a single term of corresponding form; thus,

$$\Sigma . P \cos \alpha = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c.$$
$$\Sigma . P \cos \beta = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \&c.$$
$$\Sigma . P \cos \gamma = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \&c.$$

Hence, if R be the resultant, and a, b, c the angles which its direction makes with the axes, then

$$R^2 = (\Sigma . P \cos \alpha)^2 + (\Sigma . P \cos \beta)^2 + (\Sigma . P \cos \gamma)^2,$$

which gives the value of R, and

$$\cos a = \frac{\Sigma . P \cos \alpha}{R},$$

$$\cos b = \frac{\Sigma . P \cos \beta}{R},$$

$$\cos c = \frac{\Sigma . P \cos \gamma}{R},$$

which give the values of a, b, and c.

273. *To find the conditions of equilibrium, when any number of forces act at a single point.*

Since $R^2 = X^2 + Y^2 + Z^2$, it follows that there cannot be equilibrium, or $R = 0$, unless $X = 0$, $Y = 0$, $Z = 0$; for if either of the quantities X, Y, Z have any value positive or negative, R will have some value. Hence, the required conditions of equilibrium are

$$\Sigma . P \cos \alpha = 0,$$
$$\Sigma . P \cos \beta = 0,$$
$$\Sigma . P \cos \gamma = 0.$$

If all the forces act in one plane, let the axes of x and y be

taken in that plane; then the direction of each of the forces will be at right angles with the axis of z, or γ_1, γ_2, &c. will each be $90°$; and therefore the terms $P_1 \cos \gamma_1$, $P_2 \cos \gamma_2$, &c. will each be zero. Hence the third equation will vanish. Also, α_1 and β_1, α_2 and β_2, &c. will be complementary angles. The conditions of equilibrium will consequently become in this case,

$$\Sigma . P \cos \alpha = 0,$$
$$\Sigma . P \sin \alpha = 0.$$

274. *To find the value of the resultant in terms of the components, and the angles comprised between their directions.*

In the expression $R^2 = (\Sigma . P \cos \alpha)^2 + (\Sigma . P \cos \beta)^2 + (M . P \cos \gamma)^2$ substitute for $\Sigma . P \cos \alpha$, $\Sigma . P \cos \beta$, and $\Sigma . P \cos \gamma$ their values; and since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, we have

$$\begin{aligned} R^2 &= P_1^2 + P_2^2 + P_3^2 + \&c. \\ &+ 2P_1P_2 (\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2) \\ &+ 2P_1P_3 (\cos \alpha_1 \cos \alpha_3 + \cos \beta_1 \cos \beta_3 + \cos \gamma_1 \cos \gamma_3) + \&c. \\ &+ 2P_2P_3 (\cos \alpha_2 \cos \alpha_3 + \cos \beta_2 \cos \beta_3 + \cos \gamma_2 \cos \gamma_3) + \&c. \end{aligned}$$

But if $\widehat{P_1P_2}$ denote the angle between the directions of P_1 and P_2,

$$\cos \widehat{P_1P_2} = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

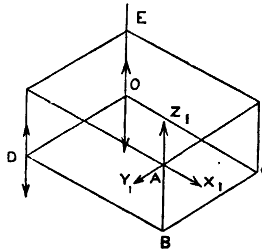
Hence,

$$\begin{aligned} R^2 = P_1^2 &+ P_2^2 + P_3^2 + \&c. \\ &+ 2P_1P_2 \cos \widehat{P_1P_2} + 2P_1P_3 \cos \widehat{P_1P_3} + \&c. \\ &+ 2P_2P_3 \cos \widehat{P_2P_3} + 2P_2P_4 \cos \widehat{P_2P_4} + \&c. \\ &+ \&c. \end{aligned}$$

275. When forces act at different points of a rigid body they are not, as with forces acting at a single point, always reducible to a single force. Under a certain condition only, to be presently deduced, is this possible. They can, however, be always reduced to a single force and a single couple.

276. *To find the resultant force and the resultant couple, when any number of forces act at various points of a rigid body.*

Let P_1, P_2, &c. be the forces. Let α_1, β_1, γ_1, α_2, β_2, γ_2, &c. be the angles which the forces make severally with any three rectangular axes, and x_1, y_1, z_1, x_2, y_2, z_2, &c. the co-ordinates of their several points of application.



Resolve P_1 into three forces acting in directions parallel to the axes; these will be $P_1 \cos \alpha_1$, $P_1 \cos \beta_1$, and $P_1 \cos \gamma_1$.

Let A be the point of application of P_1, and X_1, Y_1, Z_1 the three components into which P_1 has been resolved.

Let OC, OD, OE be the axes of x, y, z respectively, and OC, CB, BA the co-ordinates of the point A; whence $OC = x_1$, $CB = y_1$, $BA = z_1$. Through B draw BD parallel to OC; then $BD = x_1$, and $DO = y_1$.

At D and at O introduce two pairs of equal and opposite forces each equal to Z_1, and remove the point of application of Z_1 from A to B. Then, instead of Z_1 at A, we have a force Z_1 acting along the axis of z; a positive couple, whose moment is $Z_1 \times DO$, or $Z_1 y_1$, acting in the plane of yz; and a negative couple, whose moment is $Z_1 \times DB$, or $Z_1 x_1$, acting in a plane parallel to xz. Let the latter couple be removed into the plane of xz, and we have, instead of Z_1 at A,

a force Z_1 acting along the axis of z,
a couple $Z_1 y_1$ acting in the plane of yz,
and a couple $-Z_1 x_1$ " " zx.

In like manner, Y_1 at A may be resolved into

a force Y_1 acting along the axis of y,
a couple $Y_1 x_1$ acting in the plane of xy,
and a couple $-Y_1 z_1$ " " yz.

And X_1 at A may be resolved into

a force X_1 acting along the axis of x,
a couple $X_1 z_1$ acting in the plane of zx,
and a couple $-X_1 y_1$ " " xy.

Whence, by adding together the moments of those couples which act in the same plane (Art. 265), the force P_1 is resolved into the forces X_1, Y_1, Z_1 along the axes and into the couples

$$\begin{array}{lll} Z_1 y_1 - Y_1 z_1 & \text{in plane of} & yz, \\ X_1 z_1 - Z_1 x_1 & \quad " & zx, \\ Y_1 x_1 - X_1 y_1 & \quad " & xy. \end{array}$$

Resolving the other forces P_2, P_3, &c., in a similar manner, we obtain, in place of the original forces, the forces

$$\begin{array}{llll} \Sigma.X & \text{acting along} & \text{the axis of} & x, \\ \Sigma.Y & \quad " & \quad " & y, \\ \Sigma.Z & \quad " & \quad " & z; \end{array}$$

and the couples

$$\begin{array}{lll} \Sigma(Zy - Yz) & \text{in plane of} & yz, \\ \Sigma(Xz - Zx) & \quad " & zx, \\ \Sigma(Yx - Xy) & \quad " & xy. \end{array}$$

Let L, M, N represent severally these couples; then, if R be the resultant force, and G the moment of the resultant couple, by Articles 267, 270,

$$R^2 = (\Sigma.X)^2 + (\Sigma.Y)^2 + (\Sigma.Z)^2,$$

and $$G^2 = L^2 + M^2 + N^2.$$

And if α, β, γ be the angles which the direction of R makes with the axes, and λ, μ, ν those which the axis of G makes with the axes,

$$\cos\alpha = \frac{\Sigma.X}{R}, \quad \cos\beta = \frac{\Sigma.Y}{R}, \quad \cos\gamma = \frac{\Sigma.Z}{R},$$

$$\text{and } \cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

277. *To find the resultant of any number of forces acting at various points of a rigid body, when all the forces act in the same plane.*

Let the plane of the forces be the plane of xy, then γ_1, γ_2, &c., will each $= 90°$, and z_1, z_2, &c., will each $= 0$. Also, $\alpha_1 + \beta_1 = 90°$, $\alpha_2 + \beta_2 = 90°$, &c.

Hence the given forces, resolved as in the preceding Article, will be reduced to forces $\Sigma.X$ and $\Sigma.Y$ acting along the axes of x and y respectively, and a couple whose moment is G, the values of $\Sigma.X$, $\Sigma.Y$, and G, being those given by the following equations:—

$$\Sigma.X = \Sigma.P\cos\alpha,$$
$$\Sigma.Y = \Sigma.P\sin\alpha,$$
$$G = \Sigma(Yx - Xy).$$

If $\Sigma.X$ and $\Sigma.Y$ each vanish, the resultant of all the forces will be the couple whose moment is G. If $\Sigma.X$ and $\Sigma.Y$ do not both vanish, their resultant and the resultant couple may be combined into a single force, which will consequently be the resultant of the given forces. Let R be this resultant, α the angle its direction makes with the axis of x, and x, y the co-ordinates of any point in its direction. Resolve R into forces acting along the axes and a couple, these must be equal to $\Sigma.X$, $\Sigma.Y$, and G respectively. Hence,

$$R\cos\alpha = \Sigma.X, \qquad R\sin\alpha = \Sigma.Y,$$

and $$R(x\sin\alpha - y\cos\alpha) = G.$$

Whence $$R^2 = (\Sigma.X)^2 + (\Sigma.Y)^2,$$

and $$x(\Sigma.Y) - y(\Sigma.X) = G.$$

The former of these equations determines the magnitude of R, and the latter the line of its direction.

278. *When any number of forces act at various points of a rigid body, to find the condition that they may have a single resultant.*

It has already been shown, that when any number of forces act at various points of a rigid body, the resultant force $= \sqrt{\{(\Sigma.X)^2 + (\Sigma.Y)^2 + (\Sigma.Z)^2\}}$, and the resultant couple $= \sqrt{(L^2 + M^2 + N^2)}$.

If $\Sigma.X = 0$, $\Sigma.Y = 0$, $\Sigma.Z = 0$, there is no resultant force, but a resultant couple only; hence, in this case, the given forces do not admit of a single resultant.

If $L = 0$, $M = 0$, and $N = 0$, there is no resultant couple, and hence the given forces will be reduced to a single resultant passing through the origin.

Let neither $\Sigma . X$, $\Sigma . Y$, and $\Sigma . Z$, nor L, M, and N, vanish together, and let R be the resultant, a, b, c, the angles which its direction makes with the axes, and x, y, z, the co-ordinates of a point in its direction. Let R be resolved, in the same manner as the forces P_{11}, &c. (Art. 276), into three forces acting along the axes, and into three couples acting in the co-ordinate planes; these must be equal severally to $\Sigma . X$, $\Sigma . Y$, $\Sigma . Z$, L, M, N. Hence,

$$R \cos a = \Sigma . X, \quad R \cos b = \Sigma . Y, \quad R \cos c = \Sigma . Z.$$

$$R (y \cos c - z \cos b) = L,$$
$$R (z \cos a - x \cos c) = M,$$
$$R (x \cos b - y \cos a) = N.$$

Substituting, in the last three equations, the values of R cos a, &c., given by the first three, we obtain

$$y (\Sigma . Z) - z (\Sigma . Y) = L,$$
$$z (\Sigma . X) - x (\Sigma . Z) = M,$$
$$x (\Sigma . Y) - y (\Sigma . X) = N.$$

Multiplying both sides of the first equation by $\Sigma . X$, of the second by $\Sigma . Y$, and of the third by $\Sigma . Z$, and adding together, x, y, and z will be eliminated. The resulting equation will be

$$L (\Sigma . X) + M (\Sigma . Y) + N (\Sigma . Z) = 0,$$

which is the condition required.

279. *To find the conditions of equilibrium, when any number of forces act at various points of a rigid body.*

In order that there may be equilibrium, the resultant force and the resultant couple must both vanish. But that R may vanish, $\Sigma . X$, $\Sigma . Y$, $\Sigma . Z$, must each vanish; and that G may vanish, L, M, and N must each vanish. Hence, the six conditions of equilibrium are

$$\Sigma . P \cos \alpha = 0,$$
$$\Sigma . P \cos \beta = 0,$$
$$\Sigma . P \cos \gamma = 0.$$
$$\Sigma . P (y \cos \gamma - z \cos \beta) = 0,$$
$$\Sigma . P (z \cos \alpha - x \cos \gamma) = 0,$$
$$\Sigma . P (x \cos \beta - y \cos \alpha) = 0.$$

In like manner, if all the forces act in one plane, the conditions of equilibrium are three only, viz.,

$$\Sigma . \mathrm{P} \cos a = 0,$$
$$\Sigma . \mathrm{P} \sin a = 0,$$
$$\Sigma . \mathrm{P} (x \sin a - y \cos a) = 0.$$

280. *To find the conditions of equilibrium of forces acting at various points of a rigid body, when one point in the body is fixed.*

Let the fixed point be taken as the origin. There will evidently be equilibrium, if all the forces have a single resultant passing through the origin. Hence, by Art. 278, the conditions required are that

$$\mathrm{L} = 0, \qquad \mathrm{M} = 0, \qquad \mathrm{N} = 0.$$

If all the forces act in one plane, the only condition (Art. 277) is that $\mathrm{G} = 0$, or that

$$\Sigma . \mathrm{P} (x \sin a - y \cos a) = 0.$$

But if p_1 be the length of the perpendicular drawn from the origin to the line of direction of P_1,

$$p_1 = x_1 \sin a_1 - y_1 \cos a_1;$$

and therefore the condition of equilibrium may be written,

$$\Sigma . \mathrm{P}p = 0.$$

281. *To find the conditions of equilibrium of forces acting at various points of a rigid body, when two points in the body are fixed.*

Let the line joining the given points be taken as the axis of z; then, since the body is fixed upon this axis, the forces acting along this line cannot produce motion, nor can the forces acting along the axes of x and y, since they tend only to press the body against the fixed axis; nor can the couples in the planes of yz and zx produce any motion. Hence, the only condition of equilibrium is that

$$\Sigma . \mathrm{P} (x \cos \beta - y \cos a) = 0.$$

If the body were free to slide along the axis of z, then, in addition to the preceding condition, the forces acting along the axis of z must also vanish, or

$$\Sigma . P \cos \gamma = 0.$$

EXAMPLES.

1. If three forces acting upon a rigid body at any points be in equilibrium, show from the general equations that they must lie in the same plane.

2. Show that any number of forces, not in equilibrium, acting in different directions, at various points of a rigid body, may be always reduced to *two* single forces.

3. If R be the resultant of the forces P_1, P_2, &c., acting in one plane, at various points of a rigid body, show that

$$\cos \widehat{RP_1} = \frac{P_1 \cos \widehat{P_1P_1} + P_2 \cos \widehat{P_1P_2} + P_3 \cos \widehat{P_1P_3} + \&c.}{R},$$

the angle $\widehat{P_1P_1}$ (which, of course, $= 0$) being introduced for the sake of symmetry.

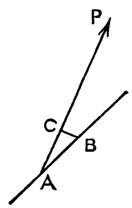
4. Show that the preceding expression holds good, if sines be written for cosines.

CHAPTER III.

ON VIRTUAL VELOCITIES.

282. Let A, the point of application of a force P, be displaced from A to B. Draw BC perpendicular to the direction of P, then the distance AC is termed the *virtual velocity* of the point A with respect to the force P.

The virtual velocity is taken to be positive when, as in the figure, the perpendicular falls on the same side of the point of application as the point *towards* which the force acts. If the perpendicular falls on the opposite, the virtual velocity is reckoned negative. If the direction of P make an angle α with the line AB, the virtual velocity is equal to



$$AB \cos \alpha.$$

283. *If any number of forces acting upon a point be in equilibrium, and the point of application be displaced, the sum of the products of each force into its virtual velocity is equal to zero.*

Let P_1, P_2, &c., be the given forces, and v_1, v_2, &c., their virtual velocities; then shall $P_1 v_1 + P_2 v_2 + \&c. = 0$.

Let α_1, α_2, &c., be the angles which the directions of the forces make with the line joining the two positions of the point of application. Then, if this line be taken as the axis of x, the conditions of equilibrium give

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0.$$

Let a be the distance between the two positions of the given point, then

$$P_1 \, a \cos \alpha_1 + P_2 \, a \cos \alpha_2 + \&c. = 0.$$

But (Article 282) $v_1 = a \cos \alpha_1$, $v_2 = a \cos \alpha_2$, &c.

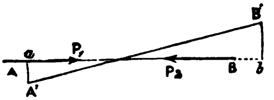
$$P_1 v_1 + P_2 v_2 + \&c. = 0;$$

or, employing the notation explained in the preceding chapter,

$$\Sigma . Pv = 0.$$

284. *If two equal and opposite forces act at two points in a rigid body, and these points be displaced through an indefinitely small space, the virtual velocities of the forces shall be equal in magnitude, and of opposite signs.*

Let P_1, P_2, be equal and opposite forces acting at the points A and B in a rigid body. Let the line AB be supposed to lie in the plane of the paper. Let A be taken as the origin of co-ordinates, and the line AB as the axis of x; and let the distance $AB = c$.



Let A and B be displaced through indefinitely small spaces to the points A′ and B′, situated either within or without the plane of the paper. Let x_1, y_1, z_1, be the co-ordinates of A′, and $c + x_2$, y_2, z_2, those of B′; x_1, y_1, z_1, x_2, y_2, z_2, being indefinitely small quantities. Then $x_1 = Aa$, or the virtual velocity of P_1 and $x_2 = Bb$, or the virtual velocity of P_2.

Since the points A, B, are rigidly connected, $A'B' = AB = c$;

$$\therefore \quad c^2 = (c + x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.*$$

But since x_1, y_1, z_1, x_2, y_2, z_2, are indefinitely small quantities, their squares and the products of any two of them may be disregarded. Rejecting all such terms, the equation becomes

$$c^2 = c^2 + 2cx_2 - 2cx_1;$$

$$\therefore \quad 0 = 2cx_2 - 2cx_1,$$

$$\text{or,} \quad x_1 = x_2.$$

* If a_1, b_1, c_1, a_2, b_2, c_2, be the co-ordinates of two points, the square of the distance between them $= (a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2$. Let A, fig. Art. 276, be the point whose co-ordinates are a_1, b_1, c_1, and O the point whose co-ordinates are a_2, b_2, c_2. The distance AO is the diagonal of a right solid whose sides are severally $a_1 - a_2$, $b_1 - b_2$, $c_1 - c_2$, and therefore

$$AO^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2.$$

Hence the virtual velocities of P_1 and P_2 are equal in magnitude. Also, x_1 is measured along the direction of P_1, and is therefore positive; but x_2 is not measured along the direction of P_2, and is therefore negative; the virtual velocities, therefore, have contrary signs.

285. *If any number of forces acting at various points of a rigid body be in equilibrium, and if the body receive an indefinitely small disturbance, the sum of the products of each force into its virtual velocity is equal to zero.*

Let P_1, P_2, &c. be the forces, A_1, A_2, &c. their points of application, and v_1, v_2, &c. their virtual velocities; it is required to show that $P_1v_1 + P_2v_2 + \&c. = 0$.

Since A_1, A_2, &c. are points in a rigid body, we may suppose them connected by rigid rods. Let $Q_{2\cdot 1}$ denote the force exerted on A_1, from its connection with A_2, and $Q_{1\cdot 2}$ the force exerted on A_2, from its connection with A_1. These are evidently equal forces, and act along the line AA_2. In like manner, let $Q_{3\cdot 1}$ $Q_{1\cdot 3}$ denote the forces acting along A_1A_3, $Q_{2\cdot 3}$ $Q_{3\cdot 2}$ those along A_2A_3, and so on.

If, then, instead of the rods, we suppose the forces $Q_{1\cdot 2}$, $Q_{1\cdot 3}$, $Q_{2\cdot 3}$, &c. to act at the various points of the given body, the forces acting at each point must severally be in equilibrium.

Let $v_{1\cdot 2}$ be the virtual velocity of $Q_{1\cdot 2}$, $v_{2\cdot 3}$ that of $Q_{2\cdot 3}$, and so on. Then, by Art. 283,

$$P_1v_1 + Q_{2\cdot 1}\, v_{2\cdot 1} + Q_{3\cdot 1}\, v_{3\cdot 1} + \&c. = 0,$$
$$P_2v_2 + Q_{1\cdot 2}\, v_{1\cdot 2} + Q_{3\cdot 2}\, v_{3\cdot 2} + \&c. = 0,$$
$$P_3v_3 + Q_{1\cdot 3}\, v_{1\cdot 3} + Q_{2\cdot 3}\, v_{2\cdot 3} + \&c. = 0.$$
$$\&c. \qquad\qquad \&c.$$

But since A_1, A_2, &c. have been displaced through an indefinitely small space, it follows from the preceding proposition that $v_{2\cdot 1} = -v_{1\cdot 2}$, $v_{2\cdot 3} = -v_{3\cdot 2}$, &c. Hence, in the foregoing series of equations, for each term arising from the mutual action of the points of the body there will be a corresponding term, equal in

magnitude but of contrary sign. Therefore, if the several equations be added together, these terms will mutually destroy each other. Whence,

$$P_1 v_1 + P_2 v_2 + P_3 v_3 + \&c. = 0.$$

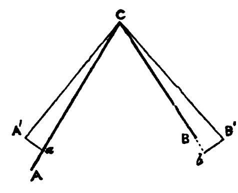
286. It will be seen, from Art. 282, that when the line joining the two positions of the point of application forms a right angle with the direction of the force, the virtual velocity is equal to zero. Hence, the virtual velocities of all resistances, when bodies rest upon a perfectly smooth immoveable plane, are severally equal to zero. Also, the virtual velocities of all reactions at fixed points must, it is plain, be severally equal to zero. Hence, when forces acting upon a rigid body are in equilibrium, some of which are resistances of smooth planes and reactions of fixed points, these latter may be disregarded, and the principle of virtual velocities holds good of the remaining forces.

287. *If two material particles, in equilibrium under any forces, be connected by an inextensible cord, the virtual velocities of the tensions in the cord are equal in magnitude, but have opposite signs.*

This is evidently true when the displacement takes place in the direction of the cord itself.

If the displacement do not take place in the direction of the cord:—first, let the cord form a single straight line; then, since the particles preserve an invariable distance, the cord being supposed inextensible, the proposition is identical with that already proved in Art. 284.

Secondly, let the cord pass round a fixed point so as to form two straight lines. Let ACB be the original position of the cord. Let A and B be displaced over an indefinitely small space to A′ and B′, the positions of A′ and B′ being such as to allow the cord to remain stretched. Then A′C + CB′ = AC + CB. With C as the centre, and CA′, CB′ as



radii, describe the small arcs $A'a$, $B'b$. Since the displacement is indefinitely small, these arcs may be regarded as straight lines perpendicular to AC and BC, and therefore Aa and Bb are the virtual velocities of A and B. But $AC = A'C + Aa$, and $BC = B'C - Bb$;

$$\therefore \quad AC + BC = A'C + B'C + Aa - Bb;$$

$$\therefore \quad o = Aa - Bb,$$

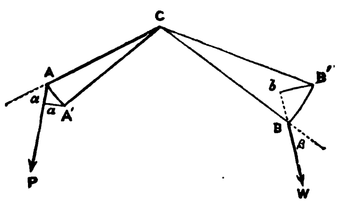
$$\text{or} \quad Aa = Bb.$$

Hence, the virtual velocities are equal in magnitude. The tension on A acts towards C, therefore Aa falling along the direction is positive. The tension on B acts towards C, and therefore Bb is negative. Hence, the virtual velocities have contrary signs.

288. From the preceding Articles, it follows that if a system of rigid bodies, connected by flexible inextensible cords, be in equilibrium under any forces, the principle of virtual velocities applies to the forces irrespective of the tensions in the cords, since the products arising from these enter in pairs which mutually destroy each other, and (Art. 286) irrespective also of the resistance of smooth planes and the reactions of fixed points.

289. *To verify the principle of virtual velocities in the bent lever.*

Let ACB be the lever, and C its fulcrum. Let the forces P and W make the angles α and β respectively with the arms produced. Then the condition of equilibrium is (Art. 77) that $P \cdot AC \sin \alpha = W \cdot BC \sin \beta$. Let $A'CB'$ be the position of the lever after displacement. Since the displacement is indefinitely small, AA' and BB' may be regarded as straight lines perpendicular to AC and BC respectively, and therefore the angle $A'Aa = 90° - \alpha$,



and the angle $B'Bb = 90° - \beta$. By the principle of virtual velocities,

$$P \cdot Aa - W \cdot Bb = 0.$$

But $\quad Aa = AA' \cos A'Aa = AA' \sin \alpha,$

and $\quad Bb = BB' \cos B'Bb = BB' \sin \beta;$

$\therefore \quad P \cdot AA' \sin \alpha = W \cdot BB' \sin \beta,$

or $\quad P \sin \alpha : W \sin \beta :: BB' : AA';$

but since the triangles ACA', BCB' are similar,

$$BC : AC :: BB' : AA',$$

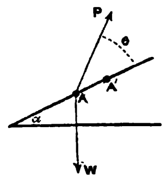
$\therefore \quad P \sin \alpha : W \sin \beta :: BC : AC,$

which is identical with the condition of equilibrium stated above.

290. *To verify the principle of virtual velocities in the inclined plane.*

Let a weight W be sustained on an inclined plane at the point A by a power P, whose direction makes an angle θ with the plane. Let α be the inclination of the plane. Then, by Art. 105,

$$P : W :: \sin \alpha : \cos \theta.$$



Let A be displaced to A′, then since the direction of P makes an angle θ with AA′ (Art. 282),

$$P\text{'s virtual velocity} = AA' \cos \theta;$$

and since the direction of W makes an angle $90° - \alpha$ with AA′,

$$W\text{'s virtual velocity} = AA' \cos (90° - \alpha) = AA' \sin \alpha,$$

and not falling along the direction of the force is negative. Therefore, by the principle of virtual velocities,

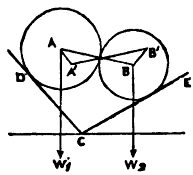
$$P \cdot AA' \cos \theta - W \cdot AA' \sin \alpha = 0;$$

$$\therefore \quad P \cos \theta = W \sin \alpha,$$

the same relation as that given above.

291. *To apply the principle of virtual velocities to determine the position of equilibrium of two spheres, of uniform density, resting upon two smooth inclined planes.*

Let A, B be the centres of two spheres whose weights are W_1, W_2 respectively, resting on the smooth inclined planes CD, CE, whose inclinations are α_1, α_2. In the position of equilibrium, let θ be the angle which the line AB makes with the vertical line through A. Let the system receive an indefinitely small displacement, in consequence of which A is moved to A′, and B to B′. The lines AA′, BB′ are parallel respectively to CD and CE; therefore the angle between AA′ and the direction of $W_1 = 90^\circ - \alpha_1$, and the angle between BB′ and the direction of $W_2 = 90^\circ - \alpha_2$. Hence,



$$\text{virtual velocity of } W_1 = AA' \sin \alpha_1,$$
$$\text{,, \quad ,, \quad } W_2 = BB' \sin \alpha_2;$$

the former is positive and the latter negative. Therefore, by the principle of virtual velocities,

$$W_1 \,.\, AA' \sin \alpha_1 - W_2 \,.\, BB' \sin \alpha_2 = 0. \qquad \text{(i.)}$$

But since the distance between A and B is invariable, the virtual velocities of A and B in the direction of AB are equal in magnitude and have contrary signs (Art. 284);

$$\therefore \qquad AA' \cos A'AB = -BB' \cos B'BA.$$

But the angle $A'AB = \theta - (90^\circ - \alpha_1) = \theta + \alpha_1 - 90^\circ$; and the angle $B'BA = \theta + 90^\circ - \alpha_2 = \theta - \alpha_2 + 90^\circ$;

$$\therefore \qquad AA' \sin(\theta + \alpha_1) = BB' \sin(\theta - \alpha_2). \qquad \text{(ii.)}$$

Eliminating AA′ and BB′ between the equations i. and ii. we obtain

$$W_1 \sin \alpha_1 \sin(\theta - \alpha_2) = W_2 \sin \alpha_2 \sin(\theta + \alpha_1).$$

Divide both sides of this equation by $\sin \alpha_1 \sin \alpha_2 \cos \theta$, then

$$W_1 (\tan\theta \cot \alpha_2 - 1) = W_2 (\tan\theta \cot \alpha_1 + 1);$$

$$\therefore \qquad \tan \theta = \frac{W_1 + W_2}{W_1 \cot \alpha_2 - W_2 \cot \alpha_1}.$$

292. The method pursued in the preceding will serve also to determine the position of equilibrium of a bar resting on two smooth inclined planes. For, let AB be the bar, and AA′, BB′ portions of the planes on which it is resting. The inclinations of these planes are α_1 and α_2 respectively. Let W be the weight of the bar, resolve W acting at the centre of gravity of the bar into two forces W_1 and W_2, acting at A and B respectively. Then, as in the preceding,

$$\tan\theta = \frac{W_1 + W_2}{W_1 \cot\alpha_2 - W_2 \cot\alpha_1}.$$

Let a and b be the distance of the centre of gravity of the bar from the points A and B respectively, then

$$W_1 = W \cdot \frac{b}{a+b}, \text{ and } W_2 = W\frac{a}{a+b}.$$

Substituting these values in the expression just given, and

$$\tan\theta = \frac{a+b}{b\cot\alpha_2 - a\cot\alpha_1}.$$

Or, if ϕ be the angle which the beam, in the position of equilibrium, makes with the horizontal line, since $\phi = 90° - \theta$,

$$\tan\phi = \frac{b\cot\alpha_2 - a\cot\alpha_1}{a+b}.$$

293. *If any system of rigid bodies is in equilibrium under the action of no forces but their weights, tensions in inextensible cords, resistances of smooth immoveable planes, and the reactions of fixed points, and if an indefinitely small motion be communicated to its parts, the centre of gravity will neither ascend nor descend.*

Let P_1, P_2, &c. be the weights of the several particles of the system, and z_1, z_2, &c. the distances of their centres of gravity from any horizontal plane; then, if h be the distance of the centre of gravity from the same plane (Art. 69),

$$h = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \text{\&c.}}{P_1 + P_2 + P_3 + \text{\&c.}}.$$

Let the system receive an indefinitely small displacement, and let h, z_1, z_2, &c. become h', $z_1 + v_1$, $z_2 + v_2$, &c.; then

$$h' = \frac{P_1(z_1 + v_1) + P_2(z_2 + v_2) + P_3(z_3 + v_3) + \&c.}{P_1 + P_2 + P_3 + \&c.};$$

$$\therefore \qquad h' - h = \frac{P_1 v_1 + P_2 v_2 + P_3 v_3 + \&c.}{P_1 + P_2 + {}_3 + \&c.}.$$

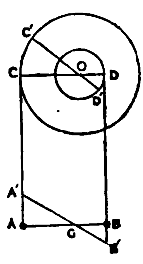
But, by Art. 288, the principle of virtual velocities is applicable to the external forces alone, irrespective of the tensions, resistances, &c. Therefore,

$$P_1 v_1 + P_2 v_2 + P_3 v_3 + \&c. = 0;$$

$$\therefore \qquad h' - h = 0; \text{ or } h' = h.$$

294. *To verify the preceding theorem in the wheel and axle and the double inclined plane.*

First: Let P and W be in equilibrium in the wheel and axle. Let R be the radius of the wheel, and r the radius of the axle, and let A, B be the respective positions of P and W when in the same horizontal line. The centre of gravity of P and W is evidently in the line AB. Let the machine move on its axis through an angle COC′, in consequence of which A moves to A′, and B to B′. Then AA′ is equal to the arc CC′, and BB′ to the arc DD′. But CC′, DD′ are similar arcs of circles, and are therefore as the radii, or



$$CC' : DD' :: R : r;$$

$$\therefore \qquad AA' : BB' :: R : r.$$

$$\text{But} \qquad P : W :: r : R;$$

$$\therefore \qquad P : W :: BB' : AA'.$$

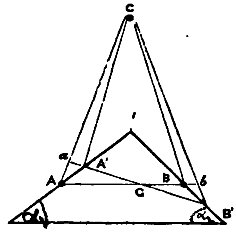
Also, since the triangles AGA′, BGB′ are similar,

$$BB' : AA' :: B'G : A'G;$$

$$\therefore \qquad P : W :: B'G : A'G,$$

or G, a point in the horizontal line AB, is the centre of gravity of P and W in their new position.

Secondly: Let the weights P and W, connected by a cord passing over a fixed pulley at C, be in equilibrium upon a double inclined plane. Let α_1, α_2 be the inclinations of the planes, and β_1, β_2 the angles made by the cord with the planes. Let T be the tension in the cord; then, since T, P, and the resistance at A are in equilibrium, by Art. 105,



$$P : T :: \cos\beta_1 : \sin\alpha_1.$$

Similarly, $$W : T :: \cos\beta_2 : \sin\alpha_2;$$

$\therefore$ $$P : W :: \cos\beta_1 \sin\alpha_2 : \cos\beta_2 \sin\alpha_1.$$

Let A and B be the positions of P and W when in the same horizontal line. Let A and B be moved through indefinitely small spaces to A′ and B′. With C as a centre, describe the arcs A′*a*, B*b*. Then A*a* and B′*b*, being the portions by which the two parts of the cord are severally shortened and lengthened, are equal. The arcs A′*a*, B*b* being indefinitely small, may be regarded as straight lines perpendicular to AC and B′C respectively; that is, $Aa = AA' \cos\beta_1$, and $B'b = BB' \cos\beta_2$;

$\therefore$ $$AA' \cos\beta_1 = BB' \cos\beta_2.$$

Let the angle $AGA' = \theta$, then

$$A'G = AA' \frac{\sin\alpha_1}{\sin\theta},$$

and $$B'G = BB' \frac{\sin\alpha_2}{\sin\theta},$$

$\therefore$ $$\frac{A'G}{B'G} = \frac{AA' \sin\alpha_1}{BB' \sin\alpha_2} = \frac{\cos\beta_2 \sin\alpha_1}{\cos\beta_1 \sin\alpha_2}.$$

$\therefore$ $$A'G : B'G :: W : P,$$

or G is the centre of gravity of P at A′ and W at B′.

295. The importance of the principle of virtual velocities has led to attempts to supply a demonstration of it independently of the equations of equilibrium deduced in the preceding chapter,

but none of these are sufficiently elementary in their character to be introduced here. If, however, this principle be assumed, the conditions of equilibrium may be deduced from it.

296. *To deduce the conditions of equilibrium of a rigid body from the principle of virtual velocities.*

Every possible motion of which any rigid body is susceptible may be resolved into a motion of translation and a motion of rotation round a fixed axis.

First, let the body receive a motion of translation, such that all the points in it move over a distance m in the direction of the line which makes with the axes angles whose cosines are a, b, c, respectively. Then, if x, y, z, be the co-ordinates of the point of application of any one of the forces P, and α, β, γ, the angles which the direction of P makes with the axes; and if v be the virtual velocity of P,

$$v = m(a \cos \alpha + b \cos \beta + c \cos \gamma).$$

By the principle of virtual velocities,

$$\Sigma . \mathrm{P}v = 0;$$

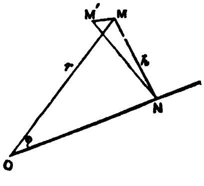
therefore,

$$a\,\Sigma . \mathrm{P} \cos \alpha + b\,\Sigma . \mathrm{P} \cos \beta + c\,\Sigma . \mathrm{P} \cos \gamma = 0.$$

But this result is independent of the values of a, b, and c, for the direction of the axes is altogether arbitrary. Consequently, each term must separately vanish, or we must have

$$\left.\begin{aligned} \Sigma . \mathrm{P} \cos \alpha = 0, \\ \Sigma . \mathrm{P} \cos \beta = 0, \\ \Sigma . \mathrm{P} \cos \gamma = 0. \end{aligned}\right\} \quad \text{(i.)}$$

Again, let the body revolve through a small angle θ about a line passing through the origin, and making with any three rectangular axes angles whose cosines are a, b, c, respectively. Let ON represent this line. Let M be the point of application of any one of the forces, and x, y, z, its co-ordinates, r its distance from the origin, and h its perpendicular distance from



ON. Let M revolve about the axis ON, through a small angle θ into the position M.′ The small arc MM′ may be regarded as a straight line perpendicular to the plane MON. Let the angles which MM′ makes with the axes have for their cosines A, B, C respectively, and let the force acting at M make with the axes the angles α, β, γ. Then, if v be the virtual velocity,

$$v = \mathrm{MM}'(\mathrm{A}\cos\alpha + \mathrm{B}\cos\beta + \mathrm{C}\cos\gamma).$$

The angles which OM makes with the axes have for their cosines $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, respectively. Hence, since MM′ is perpendicular to OM,

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0.$$

Similarly, because the angle between MM′, and a line through M parallel to ON, is a right angle,

$$\mathrm{A}a + \mathrm{B}b + \mathrm{C}c = 0.$$

From these two equations, it follows that

$$\mathrm{A} : \mathrm{B} : \mathrm{C} :: bz - cy : cx - az : ay - bx.^*$$

But $\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = 1$; whence, letting K stand as an abbreviation for

$$\sqrt{(bz-cy)^2 + (cx-az)^2 + (ay-bx)^2},$$

we obtain

$$\mathrm{A} = \frac{bz-cy}{\mathrm{K}}, \qquad \mathrm{B} = \frac{cx-az}{\mathrm{K}}, \qquad \mathrm{C} = \frac{ay-bx}{\mathrm{K}}.$$

Again, $\mathrm{MM}' = h\theta$. But if the angle $\mathrm{MON} = \phi$,

$$h = r\sin\phi,$$

and $$\cos\phi = \frac{ax+by+cz}{r};$$

* The student will find it useful to remember the following result, which he may easily deduce. Given two equations between three quantities, x, y, z, of the form

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0;$$

then $x : y : z :: b_1c_2 - b_2c_1 : c_1a_2 - c_2a_1 : a_1b_2 - a_2b_1$.

whence, remembering that $a^2+b^2+c^2=1$, and that $x^2+y^2+z^2=r^2$, it results that

$$h=\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2},$$

or,

$$h=\mathrm{K}.$$

Substituting the values of A, B, C, and MM′ just obtained in the expression $v=\mathrm{MM}'(\mathrm{A}\cos\alpha+\mathrm{B}\cos\beta+\mathrm{C}\cos\gamma)$, we have

$$v=\theta\{(bz-cy)\cos\alpha+(cx-az)\cos\beta+(ay-bx)\cos\gamma\},$$
$$=\theta\{a(y\cos\gamma-z\cos\beta)+b(z\cos\alpha-x\cos\gamma)+c(x\cos\beta-y\cos\alpha)\}.$$

By the principle of virtual velocities,

$$\Sigma\,.\,\mathrm{P}v=0.$$

Therefore,

$$\begin{aligned}&a\,\Sigma\,.\,\mathrm{P}(y\cos\gamma-z\cos\beta)\\ &+b\,\Sigma\,.\,\mathrm{P}(z\cos\alpha-x\cos\gamma)\\ &+c\,\Sigma\,.\,\mathrm{P}(x\cos\beta-y\cos\alpha)=0.\end{aligned}$$

As before, this result is independent of the direction of the axes, and therefore of a, b, and c. Consequently, each term must vanish separately; that is, we must have

$$\left.\begin{aligned}\Sigma\,.\,\mathrm{P}(y\cos\gamma-z\cos\beta)=0,\\ \Sigma\,.\,\mathrm{P}(z\cos\alpha-x\cos\gamma)=0,\\ \Sigma\,.\,\mathrm{P}(x\cos\beta-y\cos\alpha)=0.\end{aligned}\right\}\quad\text{(ii.)}$$

Equations i. and ii. correspond with the six equations of equilibrium.

CHAPTER IV.

ON FRICTION.

297. The mutual reactions, when one solid body is in contact with another, have hitherto been regarded as acting entirely in the direction of the common normal through the point of contact. This is true only upon the supposition that the surfaces in contact are perfectly smooth. Upon such hypothesis, the reactions of the surfaces oppose in no degree the motion of one body along the other, but the slightest conceivable force acting along a common tangent through the point of contact will move the body in that direction. Since, however, the surfaces of bodies in contact are in no case perfectly smooth, the foregoing result does not obtain in practice. A force varying in amount in different cases, but of an appreciable magnitude, is found to be necessary in order to move one body along the surface of another. To this force the name of *friction* is given. The total reaction of the surfaces is therefore not in the direction of the normal, but is inclined to it at a greater or less angle. If the reaction be resolved into two forces, one normal and the other tangential to the surfaces in contact, the latter component will be the force of the friction.

298. Friction is a retarding force merely; it can destroy motion, but cannot generate it. It therefore acts upon any body only when a tendency to motion exists; that is, when the other forces acting upon the body are not in equilibrium. Also, being a retarding force, its direction is the opposite of that in which the body tends to move. Hence, if in any machine the power be

greater than is required by the conditions of equilibrium, the force of friction will act concurrently with the weight; but if less, it will act concurrently with the power. Hence, a given weight may be sustained by a less power than is required, independently of friction, but needs a greater power to move it.

299. From various experiments, made with the view of determining the laws of friction, the following general results have been obtained :—

First: *Friction is increased by time.* When two bodies have been for some time in contact, it is found that a greater force is needed to move the one over the other than to keep them moving when in motion. Thus, for instance, when the surfaces are both of oak, the force required to move one over the other, after a contact of some minutes, was found, in the experiments of M. Arthur Morin, to bear to that required to keep them moving the proportion of 62 to 48; and when a surface of iron was made to move over one of oak, the forces were as 65 to 26. The time during which the force of friction continues to increase varies in different materials. When both surfaces are wood, the maximum of friction is reached after a contact of two or three minutes; when both are metals, it is reached almost instantaneously, but when wood is placed upon metal, after a contact of several days.

Secondly: *Between the same surfaces, and under the same circumstances, friction is proportional to the pressure.* The force of friction is found to depend upon the relative disposition of the surfaces in contact; as, for instance, whether the fibres be parallel or at right angles, and also upon the state of the surfaces, whether dry or lubricated. If no change be made in the disposition or state of two surfaces in contact, it is found, when the pressure is varied, that the ratio of the friction to the pressure remains constant for the same materials. This constant ratio is termed the *coefficient of friction.* Accordingly, if μ be the coefficient of friction between any two materials, and P be the normal pressure, the friction is equal to μP.

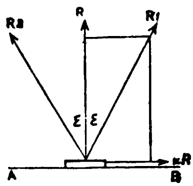
Although considerable variation exists in the values of μ, as determined by different experimentalists, the constancy of its value for the same materials may be regarded as established, both for surfaces which have been for some time in contact, and for surfaces in motion; save only that when the pressure is very large, the force of friction is found to be somewhat less than as given by this law.

Thirdly: *With the same surfaces, and under the same circumstances, friction is independent of the extent of the surface of contact.* This law again is to be regarded as only approximately true. It does not apply when the surfaces in contact are very large or very small. When the surfaces touch each other in a straight line, the friction is less than in the contact of plane surfaces of finite extent; and, on the other hand, the friction is increased when the surfaces are very large.

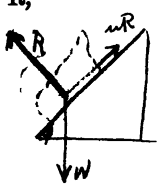
300. *Def.* The angle whose tangent is equal to μ is, for a reason that will appear presently, termed the *angle of repose.*

301. *When one body in contact with another is in a state bordering on motion, the reactions of the surfaces act in a direction which makes with the normal an angle equal to the angle of repose.*

Let a body resting upon the plane AB be upon the point of moving along the plane towards A; then, if R be the normal resistance of the plane, and μ the coefficient of friction, the force of friction will be a force μR acting towards B. Let R_1 be the resultant of R and μR; then, by the parallelogram of forces, R_1 will be represented by the diagonal of the parallelogram whose sides are proportional to R and μR. Hence, if ϵ be the angle between the directions of R_1 and R,



$$\tan \epsilon = \frac{\mu R}{R} = \mu,$$

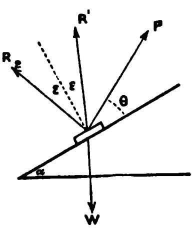


$\mu R = W \sin \phi$
$R = W \cos \phi$
$\therefore \mu = \tan \phi$

or ϵ is the angle of repose. In like manner, if the body be on the point of moving towards B, the reaction of the surfaces will act in the direction of R_2 at an angle ϵ on the other side of the normal.

302. Hence, in questions of equilibrium in which friction is considered, the reaction of two surfaces in contact may be either introduced as a single force acting at the angle of repose on either side of the normal, or as two forces R and μR acting respectively along the normal and tangent. The former will be more convenient in cases when not more than two other forces are acting upon the bodies.

303. *To find the limits of equilibrium when a body whose weight is* W *is supported upon an inclined plane by a power acting in a given direction.*



Let α be the inclination of the plane, and let the power make an angle θ with the plane.

First: Let the body be on the point of moving down the plane, the reaction of the plane will then act in the direction of R_1. Let P_1 be the value of P, in this case, then,

$$P_1 : W :: \sin \widehat{R_1 W} : \sin \widehat{R_1 P},$$

or, $$P_1 : W :: \sin(\alpha - \epsilon) : \cos(\theta + \epsilon);$$

$\therefore$ $$P_1 = W \cdot \frac{\sin(\alpha - \epsilon)}{\cos(\theta + \epsilon)}.$$

Secondly: Let the body be on the point of moving up the plane, then R_2 will be the reaction of the plane; and if P_2 be the corresponding value of P, then

$$P_2 : W :: \sin \widehat{R_2 W} : \sin \widehat{R_2 P_2},$$

or, $$P_2 : W :: \sin(\alpha + \epsilon) : \cos(\theta - \epsilon);$$

$\therefore$ $$P_2 = W \cdot \frac{\sin(\alpha + \epsilon)}{\cos(\theta - \epsilon)}.$$

304. The results just obtained may be exhibited in a slightly modified form. By the preceding,

$$P = W \cdot \frac{\sin(\alpha \mp \epsilon)}{\cos(\theta \pm \epsilon)},$$

$$\therefore \qquad P = W \cdot \frac{\sin\alpha\cos\epsilon \mp \cos\alpha\sin\epsilon}{\cos\theta\cos\epsilon \mp \sin\theta\sin\epsilon}.$$

Dividing numerator and denominator by $\cos\epsilon$, and substituting for $\tan\epsilon$ its value μ, we obtain

$$P = W \cdot \frac{\sin\alpha \mp \mu\cos\alpha}{\cos\theta \mp \mu\sin\theta}.$$

305. If the power acts horizontally, $\theta = -\alpha$, and the general result obtained in Art. 303 becomes

$$P = W \cdot \frac{\sin(\alpha \mp \epsilon)}{\cos(\alpha \mp \epsilon)},$$

Whence, $$P = W \cdot \tan(\alpha \mp \epsilon),$$

or, $$P = W \cdot \frac{\tan\alpha \mp \mu}{1 \pm \mu\tan\alpha}.$$

306. If the power acts along the plane, $\theta = 0$, and the result of Art. 303 becomes

$$P = W \cdot \frac{\sin(\alpha \mp \epsilon)}{\cos\epsilon},$$

or, $$P = W \cdot (\sin\alpha \mp \mu\cos\alpha).$$

307. *To find the inclination of the plane down which a body is on the point of sliding by its own weight.*

Let α be the inclination of the plane, then, by the preceding, if W be on the point of sliding down the plane,

$$P = W(\sin\alpha - \mu\cos\alpha).$$

But, by the hypothesis, $P = 0$;

$$\therefore \qquad \sin\alpha - \mu\cos\alpha = 0,$$

$$\therefore \qquad \tan\alpha = \mu;$$

that is, the inclination of the plane is the angle of repose. Hence, one of the methods which have been employed for determining experimentally the coefficient of friction is that of observing the inclination at which the one body is on the point of sliding over the other. The tangent of this angle gives the value of μ for surfaces that have been some time in contact.

308. *To show that the best angle of traction is the angle of repose.*

In Art. 303 it was shown, that if θ be the angle of traction, the power necessary to bring a body whose weight is W into a state bordering on a motion up a plane whose inclination is a, is

$$W\frac{\sin(a+\epsilon)}{\cos(\theta-\epsilon)}.$$

It is required to find the value of θ that will make this the least possible. Since θ enters into the denominator only, the problem will be solved by finding the value of θ, which gives the denominator its greatest possible value. The greatest value of any cosine is unity, hence θ must be taken such that

$$\cos(\theta-\epsilon)=1;$$

$$\therefore \qquad \theta-\epsilon=0,$$

$$\text{or,} \qquad \theta=\epsilon.$$

309. *To find the limits of equilibrium in the screw.*

Let W be the weight sustained by the screw, h the distance between the threads, r the radius of the cylinder, and μ the coefficient of friction.

The weight sustained by the screw may, as already seen, be regarded as a weight sustained by a horizontal power upon an inclined plane, whose height is h and base $2\pi r$; that is, upon a plane, the tangent of whose inclination is $\frac{h}{2\pi r}$. Substituting this value for tan a, in the expression given in Art. 305,

$$P=W.\frac{h \mp 2\mu\pi r}{2\pi r \pm \mu h},$$

P being supposed to act at the surface of the cylinder. If the weight be sustained by a power P′ acting at the extremity of a lever whose length is R, then

$$P' = P \cdot \frac{r}{R};$$

$$P' = W \cdot \frac{r}{R} \cdot \frac{h \mp 2\mu\pi r}{2\pi r \pm \mu h}.$$

310. *To find the limits of the ratio of the power to the normal pressures in an isosceles wedge.*

Let ABC (fig. Art. 106) be the wedge, and let the angle $ACB = 2a$. Let P, the power, act at H. Let the resistances on each side of the wedge $= \frac{1}{2}R'$, and act at the points D, E, in the directions DG, EG, making an angle ϵ (ϵ = angle of repose) with the normal, and therefore angle $90° - \epsilon$ with the sides of the wedge. Let the power be on the point of overcoming the resistance; then the angles ADG, BFG are equal to $90° - \epsilon$. Hence,

$$P : \tfrac{1}{2}R' : : \sin DGE : \sin DGH;$$

$$\therefore \quad P : \tfrac{1}{2}R' : : \sin \{180° - 2(a+\epsilon)\} : \sin \{90° + (a+\epsilon)\};$$

$$\therefore \quad P = \tfrac{1}{2}R' \frac{\sin 2(a+\epsilon)}{\cos (a+\epsilon)},$$

$$= R' \cdot \sin (a+\epsilon).$$

Secondly: Let the resistance be on the point of overcoming the power; then the angles CDG, CEG are each $= 90° - \epsilon$,

$$\therefore \quad P : \tfrac{1}{2}R' : : \sin \{180° - 2(a-\epsilon)\} : \sin \{90° + (a-\epsilon)\};$$

$$\therefore \quad P = \tfrac{1}{2}R' \frac{\sin 2(a-\epsilon)}{\cos (a-\epsilon)}.$$

$$= R' \sin (a-\epsilon).$$

If $\frac{1}{2}R$ be the normal resistance, $R = R' \cos \epsilon$. Hence,

$$P = R \cdot \frac{\sin (a \pm \epsilon)}{\cos \epsilon},$$

or,

$$P = R (\sin a \pm \mu \cos a).$$

311. *To find the greatest angle for a wedge that shall be retained in a cleft by the force of friction alone.*

By the preceding, when the resistance is on the point of overcoming the power,

$$P = R\,(\sin\alpha - \mu\cos\alpha).$$

Therefore, when $P = 0$,

$$\sin\alpha - \mu\cos\alpha = 0,$$

or,

$$\tan\alpha = \mu.$$

Hence α, or half the opening of the wedge, is equal to the angle of repose.

EXAMPLES.

1. A plane rises 5 in 13; find the greatest weight which can be sustained upon it by a power of 20 lbs. acting along the plane, the coefficient of friction being $\frac{1}{4}$.

The required weight is 130 lbs.

2. A horizontal force of 11 lbs. is on the point of drawing a weight of 13 lbs. up a plane whose inclination is equal to $\tan^{-1}(\frac{1}{3})$; determine the coefficient of friction.

The coefficient of friction is $\frac{2}{5}$.

3. The least power which will draw a weight of 205 lbs. up an inclined plane is 70 lbs., and the least power which will sustain the same weight is 20 lbs., the power in both cases acting along the plane; required the inclination of the plane, and the coefficient of friction.

The inclination of the plane is equal to

$$\sin^{-1}\left(\frac{9}{41}\right),$$

and the coefficient of friction is $\frac{1}{8}$.

4. The distance between the threads of a screw is such that a weight is just sustained without the action of any power; what is the least power which will overcome any weight W?

Let μ be the coefficient of friction, then the required power is equal to

$$\frac{2\mu W}{1-\mu^2}.$$

5. The mechanical advantage of a screw press is 240, that of the screw alone is 20; determine the power which will just overcome a resistance of 260 lbs., the coefficient of friction being $\frac{1}{3}$.

The required power is $12\frac{2}{3}$ lbs.

6. Two weights, P and Q, rest upon a double inclined plane, and are connected by a cord passing over a pulley at the summit; the coefficient of friction is the same for both planes, and P is just on the point of descending; determine the coefficient of friction.

Let α and β be the inclinations of the planes, then the coefficient of friction is equal to

$$\frac{P\sin\alpha - Q\sin\beta}{P\cos\alpha + Q\cos\beta}.$$

7. The inclination of a plane is such that a body placed upon it is upon the point of sliding down by its own weight; show that the least power which will draw the body up the plane is double the power which sustains the same weight when the plane is smooth.

CHAPTER V.

ON THE CENTRE OF GRAVITY.

312. It has been shown in the chapter on the centre of gravity in Part i., that the centre of gravity of any body can be readily found, when the centres of gravity of all its parts are known. Thus, if M_1, M_2, &c. be the magnitudes of the several parts, and x_1, x_2, &c. the distances of their centres of gravity from any plane; then, if $\bar{x}$ be the distance of the centre of gravity from the same plane,

$$\bar{x} = \frac{M_1 x_1 + M_2 x_2 + \text{\&c.}}{M_1 + M_2 + \text{\&c.}},$$

or, according to the notation used in previous chapters,

$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M}.$$

If M_1, M_2, &c. be equal to one another, and n in number, then $\Sigma . Mx = M . \Sigma x$, and $\Sigma . M = nM$, therefore, in this case,

$$\bar{x} = \frac{\Sigma . x}{n}.$$

313. Whenever a figure is symmetrical about any axis, then if of uniform density, the centre of gravity will lie in this axis, and may be found by finding its distance from any plane at right angles with the axis. When the figure is not symmetrical about any axis, the position of the centre of gravity is found by finding its distances from three planes at right angles with one another.

As before, let M_1, M_2, &c. be the magnitudes of the several parts; x_1, x_2, &c. the distances of their centres of gravity from the first plane; y_1, y_2, &c. their distances from the second plane;

and z_1, z_2, &c. their distances from the third plane: then if $\bar{x}$, $\bar{y}$, and $\bar{z}$ be the distances of the centre of gravity of the whole from the same planes,

$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M},$$

$$\bar{y} = \frac{\Sigma . My}{\Sigma . M},$$

$$\bar{z} = \frac{\Sigma . Mz}{\Sigma . M}.$$

If the given figure be a plane, it is sufficient to find the distances of the centre of gravity from two lines intersecting at right angles, and the position of the centre of gravity is given by the two equations

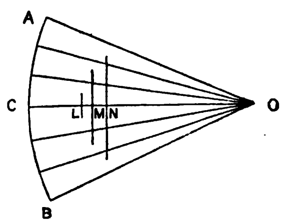
$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M},$$

$$\bar{y} = \frac{\Sigma . My}{\Sigma . M}.$$

Cor. If the centre of gravity be the origin of co-ordinates $\bar{x} = 0$, $\bar{y} = 0$, and $\bar{z} = 0$; and therefore in this case,

$$\Sigma . Mx = 0,$$
$$\Sigma . My = 0,$$
$$\Sigma . Mz = 0.$$

314. *To find the centre of gravity of the sector of a circle.*



Let r be the radius of the circle, and a the angle of the sector. Let OC bisect the angle AOB; the centre of gravity must evidently lie in this line. Let G be the centre of gravity, it remains to determine the distance GO.

Divide the angles AOC, BOC, into n equal parts. Each half of the sector will be divided into n equal sectors, and when n is very large,

these may be regarded as n equal triangles, having their centres of gravity at a distance from $O = \frac{2}{3}r$.

Join the centre of gravity of each triangle with that of the triangle similarly placed on the other side of OC. These lines will be at right angles to CO, and will be bisected by it. Let L, M, N, &c. be the points in which these lines cut OC, then the points L, M, N, &c. are the centres of gravity of each pair of triangles. Then, taking their distances from O, we have

$$x_1 = \text{LO} = \frac{2r}{3} \cos \frac{a}{4n};$$

$$x_2 = \text{MO} = \frac{2r}{3} \cos \frac{3a}{4n};$$

$$x_3 = \text{NO} = \frac{2r}{3} \cos \frac{5a}{4n};$$

&c. &c.

Hence, by Art. 312,

$$\text{GO} = \frac{\Sigma x}{n};$$

$$= \frac{2r}{3n} \left(\cos \frac{a}{4n} + \cos \frac{3a}{4n} + \cos \frac{5a}{4n} + \text{\&c.} \ldots + \cos \frac{(2n-1)a}{4n}\right).$$

$$= \frac{2r}{3} \cdot \frac{\sin \frac{1}{2}a}{2n \sin \frac{a}{4n}} *$$

* The sum of the series $\cos \phi + \cos 3\phi + \text{\&c.} + (2n-1)\,\phi$ may be easily found as follows:—

$$\sin 2\phi = 2 \cos \phi \sin \phi,$$
$$\sin 4\phi - \sin 2\phi = 2 \cos 3\phi \sin \phi,$$
$$\sin 6\phi - \sin 4\phi = 2 \cos 5\phi \sin \phi,$$
$$\text{\&c.} \qquad \text{\&c.}$$
$$\sin 2n\phi - \sin (2n-2)\,\phi = 2 \cos (2n-1)\,\phi \sin \phi.$$

Whence, by adding together,

$$\sin 2n\phi = 2 \sin \phi \left\{\cos \phi + \cos 3\phi + \text{\&c.} + \cos (2n-1)\,\phi\right\};$$

$$\therefore \qquad \cos \phi + \cos 3\phi + \text{\&c.} + \cos (2n-1)\,\phi = \frac{\sin 2n\phi}{2 \sin \phi}.$$

Since n is very large, $\frac{a}{4n}$ is very small, and therefore the angle itself may be substituted for its sine. Therefore,

$$GO = \frac{2r}{3} \cdot \frac{\sin \frac{1}{2} a}{\frac{1}{2}a};$$

$$= \frac{4r}{3} \cdot \frac{\sin \frac{1}{2} a}{a},$$

the angle a being measured by the ratio of the arc to the radius.

315. In the result just obtained, if $a = \pi$, the sector is a semicircle. Therefore, if GO denote, as before, the distance of the centre of gravity from the centre,

$$GO = \frac{4r}{3\pi}.$$

If $a = \frac{\pi}{2}$ the sector becomes a quadrant, and in this case,

$$GO = \frac{8r \sin 45^\circ}{3\pi} = \frac{4r\sqrt{2}}{3\pi}.$$

316. By a process precisely similar to that followed in Art. 314, it may be shown, that in the case of a circular arc subtending at the centre an angle a,

$$GO = \frac{2r \sin \frac{1}{2} a}{a},$$

$$= \frac{\text{rad. chord}}{\text{arc}}.$$

317. The following summations will be referred to in future Articles. They may be readily found by the method of Finite Differences.

$$1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2},$$

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

$$1^4 + 2^4 + 3^4 + \ldots\ldots + n^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30};$$

and hence, by making n as large as we please, the following equations may be made as nearly true as we please:

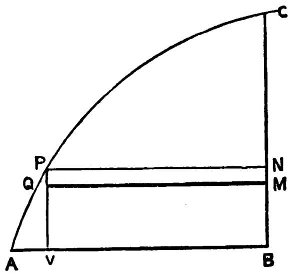
$$\frac{1 + 2 + 3 + \dots\dots + n}{n^2} = \frac{1}{2},$$

$$\frac{1^2 + 2^2 + 3^2 + \dots\dots + n^2}{n^3} = \frac{1}{3},$$

$$\frac{1^3 + 2^3 + 3^3 + \dots\dots + n^3}{n^4} = \frac{1}{4},$$

$$\frac{1^4 + 2^4 + 3^4 + \dots\dots + n^4}{n^5} = \frac{1}{5}.$$

318. *To find the centre of gravity of the area of a semi-parabola.*



Let A be the vertex and AB the axis of the parabola, and $y^2 = mx$, the equation of the curve.

Divide BC into n equal parts, n being an indefinitely large number, and let $BC = nk$.

Let MN be one of these divisions, then $MN = k$; and let $BN = pk$.

The area of the rectangle PM is equal to $k \cdot PN$, and

$$\begin{aligned} PN &= AB - AV, \\ &= \frac{BC^2}{m} - \frac{PV^2}{m}, \\ &= m^{-1}k^2(n^2 - p^2), \end{aligned}$$

$$\therefore \qquad \text{area of PM} = m^{-1}k^3(n^2 - p^2). \qquad \text{(i.)}$$

The distance of the centre of gravity of PM from the line BC is $\frac{1}{2}$ PN, or $\frac{1}{2}m^{-1}k^2(n^2 - p^2)$; therefore the product of the area of PM into the distance of its centre of gravity from BC is equal to

$$\tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2p^2 + p^4). \qquad \text{(ii.)}$$

Hence, by giving to p the values 1, 2, 3, &c., in the expression i., we obtain

$$M_1 = m^{-1}k^3(n^2 - 1^2),$$
$$M_2 = m^{-1}k^3(n^2 - 2^2),$$
$$M_3 = m^{-1}k^3(n^2 - 3^2),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$M_n = m^{-1}k^3(n^2 - n^2);$$

and therefore, taking the sum

$$\Sigma . M = m^{-1}k^3\{n^3 - (1^2 + 2^2 + 3^2 + \ldots\ldots + n^2)\},$$
$$= m^{-1} . BC^3\left(1 - \frac{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}{n^3}\right),$$
$$= \tfrac{2}{3}m^{-1}BC^3, \text{ since } n \text{ is large.}$$

In like manner, from the expression ii.,

$$M_1x_1 = \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2 . 1^2 + 1^4),$$
$$M_2x_2 = \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2 . 2^2 + 2^4),$$
$$M_3x_3 = \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2 . 3^2 + 3^4),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$M_nx_n = \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2 . n^2 + n^4);$$

and therefore

$$\Sigma . Mx = \tfrac{1}{2}m^{-2}k^5\{n^5 - 2n^2(1^2 + 2^2 + \ldots\ldots + n^2) + (1^4 + 2^4 + \ldots\ldots n^4)\},$$
$$= \tfrac{1}{2}m^{-2} . BC^5\left(1 - 2 . \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3} + \frac{1^4 + 2^4 + \ldots\ldots n^4}{n^5}\right),$$
$$= \tfrac{4}{15}m^{-2}BC^5, \text{ since } n \text{ is large.}$$

Hence, if $\bar{x}$ be the distance of the centre of gravity of the semi-parabola from BC,

$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M},$$
$$= \tfrac{2}{5}m^{-1}BC^2,$$
$$= \tfrac{2}{5}AB.$$

Again, since MN is very small, the distance of the centre of gravity of the rectangle PM from the line AB may be taken as

equal to BN or pk; therefore the product of the area of PM into the distance of the centre of gravity from AB is equal to

$$m^{-1}k^4(n^2p - p^3).$$

Giving to p the values 1, 2, 3, &c. severally, we have

$$M_1y_1 = m^{-1}k^4(n^2 . 1 - 1^3),$$
$$M_2y_2 = m^{-1}k^4(n^2 . 2 - 2^3),$$
$$M_3y_3 = m^{-1}k^4(n^2 . 3 - 3^3),$$
$$. \quad . \quad . \quad . \quad .$$
$$M_ny_n = m^{-1}k^4(n^2 . n - n^3);$$

and therefore

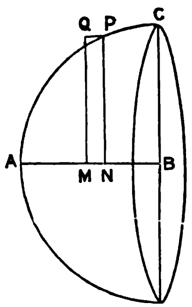
$$\Sigma . My = m^{-1}k^4\{n^2(1 + 2 + \ldots\ldots + n) - (1^3 + 2^3 + \ldots\ldots + n^3)\},$$
$$= m^{-1} . BC^4\left(\frac{1 + 2 + \ldots\ldots + n}{n^2} - \frac{1^3 - 2^3 + \ldots\ldots + n^3}{n^4}\right),$$
$$= \tfrac{1}{4}m^{-1}BC^4.$$

Hence, if $\bar{y}$ be the distance of the centre of gravity of the semi-parabola from the line AB,

$$\bar{y} = \frac{\Sigma . My}{\Sigma . M},$$
$$= \tfrac{3}{8}BC.$$

319. *To find the centre of gravity of a solid of revolution.*

Let AC be the curve which generates the solid by its revolution round the axis AB.



Let AB be divided into n equal parts, n being an indefinitely large number, and let $AB = nh$.

Let MN be one of these divisions, say the pth; then $MN = h$ and $AN = ph$.

The content of the cylinder generated by the revolution of the rectangle PM is equal to

$$\pi h . PN^2.$$

Since MN is very small, the distance of the centre of gravity of this cylinder from A may be taken as equal

to AN or ph; hence, if G be the centre of gravity of the solid,

$$GA = \frac{\Sigma . \pi h^2 p \, PN^2}{\Sigma . \pi h \, PN^2};$$

$$= \frac{\Sigma . p h^2 \, PN^2}{\Sigma . h PN^2}.$$

320. Ex. 1. To find the centre of gravity of a segment of a sphere.

Let r be the radius of the sphere, then

$$PN^2 = AN\,(2r - AN),$$

$$= 2rph - p^2h^2.$$

Hence,

$$GA = \frac{\Sigma . (2rp^2h^3 - p^3h^4)}{\Sigma . (2rph^2 \; - p^2h^3)};$$

$$= \frac{2rh^3(1^2 + 2^2 + \ldots + n^2) - h^4(1^3 + 2^3 + \ldots + n^3)}{2rh^2(1 \; + 2 \; + \ldots + n\;) - h^3(1^2 + 2^2 + \ldots + n^2)};$$

$$= \frac{2r\,AB^3 . \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3} - AB^4 . \dfrac{1^3 + 2^3 + \ldots + n^3}{n^4}}{2r\,AB^2 . \dfrac{1 \; + 2 \; + \ldots + n}{n^2} - AB^3 . \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3}};$$

$$= \frac{\frac{2}{3}r \; AB^3 - \frac{1}{4} \; AB^4}{r \; AB^2 - \frac{1}{3} \; AB^3},\text{* since } n \text{ is large};$$

$$= \frac{8r . AB - 3 . AB^2}{12\,r - 4 . AB}.$$

* From the denominator of this fraction, we see that the content of a segment of a sphere is equal to

$$\pi\,(rAB^2 - \tfrac{1}{3}AB^3).$$

Also, since $2r . AB = BC^2 + AB^2$, the content is equal to

$$\tfrac{1}{6}\,\pi\,AB\,(3BC^2 + AB^2).$$

321. COR. If the segment be a hemisphere, then $AB = r$; and therefore

$$GA = \frac{5r}{8};$$

and therefore, if O be the centre of the sphere,

$$GO = \frac{3r}{8}.$$

322. Ex. 2. To find the centre of gravity of a paraboloid.

Let m be the parameter of the generating curve, then

$$PN^2 = m \,.\, AN,$$
$$= mph.$$

Therefore,

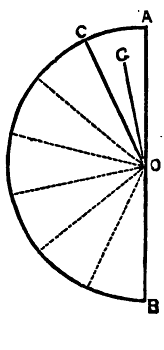
$$GA = \frac{\Sigma . mp^2h^3}{\Sigma . mph^2} = \frac{\Sigma . p^2h^3}{\Sigma . ph^2},$$

$$= \frac{h^3(1^2 + 2^2 + \ldots\ldots + n^2)}{h^2(1 + 2 + \ldots\ldots + n)},$$

$$= \frac{AB^3 . \dfrac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3}}{AB^2 . \dfrac{1 + 2 + \ldots\ldots + n}{n^2}},$$

$$= \tfrac{2}{3}AB.$$

323. *To find the centre of gravity of the portion of a sphere included between two planes intersecting in a diameter at an angle equal to any sub-multiple of two right angles.*



Let AOC be a section of the solid by a plane through the centre of the sphere, at right angles with the intersection of the planes. Then, since this plane divides the given figure symmetrically, the centre of gravity will lie in this plane.

Let the angle AOC be equal to θ, and let $n\theta = \pi$. Let G be the centre of gravity of the given solid.

Suppose the hemisphere to be divided into n equal parts, each equal and similar to the given solid. Then the sum of the moments of each part about AB is equal to the moment of the hemisphere. Hence, Art. 312,

$$GO\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}+\sin\frac{5\theta}{2}+\ldots\ldots+\sin\frac{(2n-1)\theta}{2}\right)=\frac{3nr}{8};$$

and therefore

$$GO\times\frac{1-\cos n\theta}{2\sin\frac{1}{2}\theta}=\frac{3nr}{8}.$$

But $n\theta=\pi$, and therefore $\cos n\theta=-1$. Consequently,

$$GO=\frac{3nr\sin\left(\frac{\pi}{2n}\right)}{8}.$$

324. Cor. I. Hence, generally, if θ be the angle between the planes,

$$GO=\frac{3}{8}\cdot\frac{\pi r\sin\frac{1}{2}\theta}{\theta}.$$

325. Cor. II. If the angle between the planes be indefinitely small, then, writing $\frac{\theta}{2}$ for $\sin\frac{\theta}{2}$, we have

$$GO=\frac{3\pi r}{16}.$$

326. *If the mass of each of a system of particles be multiplied by the square of its distance from a given point, the sum of these products will be the least possible when the given point is the centre of gravity of the system.*

Let m_1, m_2, &c. be the masses of the particles, r_1, r_2, &c. their distances from G, the common centre of gravity, and ρ_1, ρ_2, &c. their distances from any other point O.

Let G, the centre of gravity, be taken as the origin of co-ordinates; let x_1, y_1, z_1 be the co-ordinates of m_1; x_2, y_2, z_2, those

of m_2, and so on for the rest. Let a, b, c, be the co-ordinates of O, then

$$\rho_1^2 = (x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2;$$
$$= x_1^2 + y_1^2 + z_1^2 - 2ax_1 - 2by_1 - 2cz_1 + a^2 + b^2 + c^2,$$
$$\therefore\ m_1\rho_1^2 = m_1r_1^2 + m_1 . \mathrm{GO}^2 - 2a . m_1x_1 - 2b . m_1y_1 - 2c . m_1z_1.$$

Similarly,

$$m_2\rho_2^2 = m_2r_2^2 + m_2 . \mathrm{GO}^2 - 2a . m_2x_2 - 2b . m_2y_2 - 2c . m_2z_2,$$
$$m_3\rho_3^2 = m_3r_3^2 + m_3 . \mathrm{GO}^2 - 2a . m_3x_3 - 2b . m_3y_3 - 2c . m_3z_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence,

$$\Sigma_1 . m\rho^2 = \Sigma_1 . mr^2 + \mathrm{GO}^2 . \Sigma_1 m - 2a\Sigma_1 . mx - 2b\Sigma_1 . my - 2c\Sigma_1 . mz.$$

But by the corollary to Article 313,

$$\Sigma_1 . mx = 0, \qquad \Sigma_1 . my = 0, \qquad \Sigma_1 . mz = 0;$$

therefore,

$$\Sigma_1 . m\rho^2 = \Sigma_1 . mr^2 + \mathrm{GO}^2\Sigma_1 m,$$

that is, the sum of the products of each particle into the square of its distance from the centre of gravity is less than the sum of the products of each particle into the square of its distance from any other point O, by the quantity

$$\mathrm{GO}^2 . \Sigma_1 m;$$

or the product of the sum of the masses into the square of the distance of the point O from the centre of gravity.*

327. Cor. I. If a sphere be described, having G as its centre, and GO as its radius, then the sum of the products of each particle into the square of its distance from any point on the surface of the sphere is invariable.

328. Cor. II. If the system consist of n equal particles, $\Sigma_1 . m\rho^2 = m\Sigma_1\rho^2$, $\Sigma_1 . mr^2 = m\Sigma_1 r^2$ and $\Sigma_1 m = nm$. Therefore,

$$\Sigma_1\rho^2 = \Sigma_1 r^2 + n . \mathrm{GO}^2.$$

* If ρ_1, ρ_2, &c. denote the distances of each particle from any plane, and r_1, r_2, &c. their distances from a plane through the centre of gravity parallel to the former; then, GO being the distance between the planes, it may be easily shown that the equation

$$\Sigma . m\rho^2 = \Sigma . mr^2 + \mathrm{GO}^2 . \Sigma m$$

still obtains.

Hence, *the sum of the squares of the distances of* n *equal particles, from any point whatever, exceeds the sum of the squares of their distances from their common centre of gravity by* n *times the square of the distance of the given point from the centre of gravity.*

329. Cor. III. If three equal bodies be placed at the vertices of any triangle ABC, their centre of gravity corresponds with the centre of gravity of the triangle Hence, if G be the centre of gravity, and O any point whatever,

$$AO^2 + BO^2 + CO^2 = AG^2 + BG^2 + CG^2 + 3 \cdot GO^2.$$

330. Cor. IV. Also, if four equal weights be placed at the vertices of a triangular pyramid, their centre of gravity corresponds with the centre of gravity of the pyramid.

Hence, if A, B, C, D be the vertices of the pyramid, G its centre of gravity, and O any point whatever,

$$AO^2 + BO^2 + CO^2 + DO^2 = AG^2 + BG^2 + CG^2 + DG^2 + 4 \cdot GO^2.$$

331. *The sum of the products of each of a system of particles into the square of its distance from the centre of gravity of the system, is equal to the sum of the products of each pair into the square of their mutual distance, divided by the sum of their particles.*

Let m_1, m_2, &c. be the particles, r_1, r_2, &c. their distances from the centre of gravity of the system, and $\rho_{1.2}$, $\rho_{2.3}$, &c. the distances of m_1 and m_2, of m_2 and m_3, &c.; then will

$$\Sigma \cdot mr^2 = \frac{\Sigma \cdot m_1 \, m_2 \, \rho^2_{1\cdot 2}}{\Sigma m}.$$

or,

$$m_1 r_1^2 + m_2 r_2^2 + \&c. = \frac{m_1 m_2 \rho^2_{1.2} + m_1 m_3 \rho^2_{1.3} + \&c}{m_1 + m_2 + m_3 + \&c.}.$$

Let the centre of gravity of the particles be taken as the origin

of co-ordinates, and let $x_1, y_1, z_1, x_2, y_2, z_2$, &c. be the co-ordinates of m_1, m_2, &c.; then, by Art. 313,

$$m_1x_1 + m_2x_2 + \&c. = 0,$$
$$m_1y_1 + m_2y_2 + \&c. = 0,$$
$$m_1z_1 + m_2z_2 + \&c. = 0.$$

Square each of these, and add; then

$$\begin{aligned} &m_1^2(x_1^2 + y_1^2 + z_1^2) + m_2^2(x_2^2 + y_2^2 + z_2^2) + \&c. \\ &+ 2m_1m_2(x_1x_2 + y_1y_2 + z_1z_2) + 2m_1m_3(x_1x_3 + y_1y_3 + z_1z_3) \\ &+ \&c. = 0. \end{aligned}$$

But $r_1^2 = x_1^2 + y_1^2 + z_1^2$, and $r_2^2 = x_2^2 + y_2^2 + z_2^2$, and so on.

Also, $$\begin{aligned} \rho^2_{1\cdot2} &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2, \\ &= r_1^2 + r_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2). \end{aligned}$$

Similarly, $$\rho^2_{1\cdot3} = r_1^2 + r_3^2 - 2(x_1x_3 + y_1y_3 + z_1z_3),$$
&c. &c.

Substituting these values in the equation obtained above,

$$\begin{aligned} m_1^2r_1^2 + m_2^2r_2^2 + \&c. = m_1m_2(\rho^2_{1\cdot2} - r_1^2 - r_2^2), \\ + m_1m_3(\rho^2_{1\cdot3} - r_1^2 - r_3^2), \\ + m_2m_3(\rho^2_{2\cdot3} - r_2^2 - r_3^2), \\ + \&c. \end{aligned}$$

Hence, by transposing,

$$\begin{aligned} &m_1r_1^2(m_1 + m_2 + \&c.) \\ &+ m_2r_2^2(m_1 + m_2 + \&c.) \\ &+ \&c. \qquad = m_1m_2\rho^2_{1\cdot2} + m_1m_3\rho^2_{1\cdot3} + \&c. \end{aligned}$$

or, $$m_1r_1^2 + m_2r_2^2 + \&c. = \frac{m_1m_2\rho^2_{1\cdot2} + m_1m_3\rho^2_{1\cdot3} + \&c.}{m_1 + m_2 + m_3 + \&c.}.$$

332. Cor. I. Combining the result just obtained with that of Art. 326, we have

$$\Sigma\,.\,m\rho^2 = \frac{\Sigma\,.\,m_1m_2\rho^2_{1\cdot2}}{\Sigma m} + GO^2\,.\,\Sigma m.$$

333. Cor. II. If the system consist of n equal particles, then the result of Art. 331 becomes

$$\Sigma . \rho^2_{1 . 2} = n . \Sigma . r^2.$$

Hence, *the sum of the squares of the mutual distances of each pair out of* n *equal particles is equal to* n *times the sum of the squares of the distances of the particles from their common centre of gravity.*

334. Cor. III. Consequently, (see Art. 329,) if a, b, c be the sides of any triangle, and k, l, m the distances from the centre of gravity,

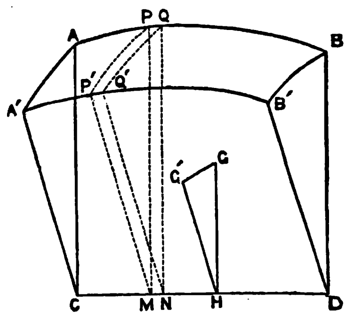
$$a^2 + b^2 + c^2 = 3 (k^2 + l^2 + m^2).$$

335. Cor. IV. Also, (see Art. 330,) if a, b, c, d, e, f be the edges of any triangular pyramid, and k, l, m, n the distances of the vertices from the centre of gravity, then

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 4 (k^2 + l^2 + m^2 + n^2).$$

GULDIN'S PROPERTIES.

336. *If any plane figure revolve about an axis lying in its own plane, the content of the generated solid is equal to the product of the generating area into the length of the arc described by its centre of gravity.*



Let ABDC be the generating area, and let it revolve about the axis CD, through any angle θ, into the position A′B′DC.

Let CD be divided into n equal parts, n being an indefinitely large number, and let $CD = nh$.

Let MN be any one of these divisions, then $MN = h$. Through M and N draw PM and QN, at right angles with CD; then PN may be regarded as a rectangle,

and the content of the solid generated by the revolution of PN is equal to

$$h \times \text{area of the sector PMP}'.$$

Let PM $= y$, then if θ be measured by the ratio of the subtended arc to the radius, the area of the sector PMP′ is equal to $\frac{1}{2}y^2\theta$; hence, the content of the solid generated by the revolution of PN is equal to

$$\tfrac{1}{2}y^2\theta h,$$

and therefore

$$\text{content of solid} = \tfrac{1}{2}\Sigma_1 . y^2\theta h;$$
$$= \tfrac{1}{2}\theta\Sigma_1 . y^2 h.$$

Again, the area of the rectangle PN is equal to yh, and the distance of its centre of gravity from CD is equal to $\frac{1}{2}y$, therefore the moment of PN about CD is equal to

$$\tfrac{1}{2}y^2 h.$$

Hence, if $\bar{y}$ be the distance of the centre of gravity of the area ABDC from CD, then

$$\bar{y} = \frac{\frac{1}{2}\Sigma_1 . y^2 h}{\Sigma_1 . yh}.$$

Therefore,

$$\text{content of solid} = \bar{y}\theta \times \Sigma_1 . yh.$$

But if G be the centre of gravity of the area ABDC, the arc GG′ $= \bar{y}\theta$ and $\Sigma_1 . yh$ is the area of ABDC; therefore,

$$\text{content of solid} = \text{GG}' \times \text{area of ABDC}.$$

337. *If any plane curve revolve about an axis lying in its own plane, the area of the generated surface is equal to the product of the length of the curve into the length of the arc described by its centre of gravity.*

Let AB be the curve, and let it revolve about the axis CD, through any angle θ, into the position A′B′.

Let AB be divided into n equal parts, n being an indefinitely large number, and let AB $= ns$.

Let PQ be one of these parts, then $PQ = s$, and since PQ is very small, it may be regarded as a straight line; and if $PM = y$, the area generated by the revolution of PQ is equal to

$$y\theta \,.\, s,$$

and hence,

$$\text{area of } ABB'A' = \Sigma \,.\, y\theta s,$$
$$= \theta\Sigma \,.\, ys.$$

Also, since PQ is small, the distance of its centre of gravity from CD may be taken as equal to y, and therefore the moment of PQ about CD is equal to

$$ys;$$

and hence, if $\bar{y}$ be the distance of the centre of gravity of AB from the line CD,

$$\bar{y} = \frac{\Sigma \,.\, ys}{AB},$$

and therefore

$$\text{area of } ABB'A' = \bar{y}\theta \,.\, AB.$$

If G be the centre of gravity of AB, then $GG' = \bar{y}\theta$, and therefore

$$\text{area of } ABB'A' = GG' \times AB.$$

EXAMPLES.

1. Find the centre of gravity of a segment of a circle; show that the distance of the centre of gravity from the centre of the circle is equal to the cube of the chord divided by 12 times the area of the segment.

2. Find the centre of gravity of the crescent formed by two arcs of equal circles, when the centre of one circle is upon the circumference of the other.

Let r be the radius, then the distance of the centre of gravity from the middle of the common chord of the two arcs is equal to

$$\frac{3\pi r}{2\pi + 3\sqrt{3}}.$$

3. Find the centre of gravity of a spherical sector.

Let the sector be divided into a spherical segment and a cone, let r be the radius of the sphere and x the thickness of the segment, then the distance of the centre of gravity from the middle of the spherical surface of the sector is equal to

$$\frac{2r+3x}{8}.$$

4. Find the centre of gravity of one-eighth of a sphere bounded by three planes at right angles to one another.

Show that the distance of the centre of gravity from the centre of the sphere is $$\frac{3r\sqrt{3}}{8}.$$

5. If a regular polygon of n sides be inscribed in a great circle of a sphere, the sum of the squares of the distances of the angular points of the polygon from any point on the sphere is equal to $2n$ times the square of the radius of the sphere.

6. The sum of the squares of the mutual distances of each pair of angular points in a regular hexagon is equal to the square of the periphery of the hexagon.

7. In a regular polygon of n sides, the sum of the squares of the distances of any one angular point from the rest is equal to $2n$ times the square of the radius of the circumscribing circle.

8. Find the centre of gravity of the portion of a sphere contained between two parallel planes, one of which passes through the centre of the sphere; the distance between the planes being equal to half the radius.

Let r be the radius of the sphere, then the distance of the centre of gravity from the centre of the sphere is equal to

$$\frac{21r}{88}.$$

9. Assuming that the area of an ellipse is the product of its semi-axes multiplied by π, find the centre of gravity of an elliptic quadrant.

Let a and b be the semi-axes, then the distance of the centre of gravity from the major-axis is equal to

$$\frac{4b}{3\pi},$$

and its distance from the minor-axis is equal to

$$\frac{4a}{3\pi},$$

10. Find the centre of gravity of the segment of a prolate spheroid cut off by a plane at right angles with the axis of revolution.

Let x be the length of the axis of the segment, and a the semi-major axis, then the distance of the centre of gravity from the vertex is equal to

$$\frac{8ax - 3x^2}{12a - 4x}.$$

11. Find the centre of gravity of the area contained between a parabolic arc AQP, and a chord AP drawn through the vertex.

Let C be the middle part of the chord AP, and let a parallel to the axis through C meet the curve in Q; show that the centre of gravity is in the line CQ, at a distance from C equal to

$$\frac{2\text{CQ}}{5}.$$

12. Apply Guldin's Properties to the determination of the content and surface of a ring.

Let a be the radius of the generating circle, and b the distance of its centre from the axis of revolution, then the content of the ring is equal to

$$2\pi^2a^2b,$$

and the surface of the ring is equal to

$$4\pi^2ab.$$

13. Find, by Guldin's Properties, the content of an oblate spheroid.

Let a and b be the semi-axes of the generating ellipse, the content required is equal to

$$\frac{4\pi a^2b}{3}.$$

14. The surface of a sphere whose radius is r is equal to $4\pi r^2$; determine the centre of gravity of a semi-circular arc.

15. Find the content of the solid generated by the revolution of a triangle about one of its sides.

Let a, b, and c be the sides of the triangle, and let c be the axis of revolution, then the required content is equal to

$$\frac{4\pi s\,.\,(s-a)\,(s-b)\;s-c)}{3c}.$$

16. Show that the content of the solid generated by the revolution of a segment of a circle about the diameter which is parallel to its chord is equal to

$$\frac{\pi}{6} \times (\text{chord})^3.$$

17. Find the content of the spindle generated by the revolution about its chord of a circular segment whose height is equal to half the radius.

Let r be the radius of the segment, then the required content is equal to

$$\pi r^3 \left(\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right).$$

18. Find the area of a spherical zone.

Let r be the radius of the sphere, and c the axis of the zone, then the required area is equal to

$$2\pi cr.$$

19. Assuming the preceding, show that the centre of gravity of any spherical zone or of the surface of any spherical segment is the middle point of the axis.

20. Show, from Guldin's Properties, that the content of a paraboloid of revolution is equal to the area of its plane surface multiplied by half its axis.

(For the area of a semi-parabola, refer to Art. 318.)

CHAPTER VI.

STATICAL PROBLEMS.

338. If three forces, not parallel to each other, acting at any points in a rigid body are in equilibrium, they will lie in the same plane. There must therefore be a point in which their directions, if produced, will meet. For let the directions of any two be produced till they meet, then their resultant must pass through this point, and since there is equilibrium, the third force must also pass through this point. Consequently, the triangle of forces will apply to three forces in equilibrium, acting at any points in a rigid body.

The triangle of forces will give two of the three equations of equilibrium. The third may be obtained by taking the moments of any two of the forces about any point in the direction of the third force; and since the resultant of the two must be equal and opposite to the third, these moments must be equal.

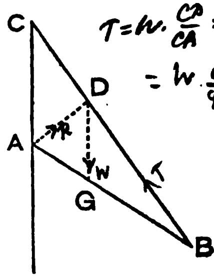
339. When more than three forces act at various points of a rigid body, many problems of equilibrium may be conveniently solved, if all the forces can be brought to act at any two points in the body. For then, since there is equilibrium, the resultant of one set of forces must be equal and opposite to the resultant of the other set, and consequently the resultant of each set must act along the line joining the two points. And if at each point a force equal and opposite to the resultant be introduced, equilibrium will subsist between each set of forces separately.

340. *One extremity of a beam* AB *is fastened by a pivot at* A,

and a cord tied to the other extremity B *is fastened to a point* C, *lying above* A *in the same vertical line, required the tension in the string.*

Let W be the weight of the beam, acting at its centre of gravity G. Let $AB = l$, $AG = a$, $AC = h$, and $BC = c$.

The three forces acting upon the beam are the weight of the beam, the tension of the string, and the resistance of the pivot. Through G draw the vertical line GD. Join AD, and let T be the tension of the string. T acts along CB, and W along GD, consequently the resultant of T and W must pass through D. But since there is equilibrium, this resultant must be equal and opposite to the third force, viz., the resistance of the pivot, and consequently AD is the direction in which this resistance acts.



$T = W \cdot \frac{CD}{CA} = \frac{W}{CA} \cdot CB \cdot \frac{AG}{AB} = W \cdot \frac{ac}{hl}$.

The sides of the triangle ACD are severally parallel to the directions of the three forces,

$$\therefore \qquad T : W :: CD : AC,$$

$$\text{or,} \qquad T = W\frac{CD}{h}.$$

But since GD and AC are parallel, CD : CB :: AG : AB, or

$$CD : c :: a : l,$$

$$\therefore \qquad CD = \frac{ac}{l},$$

$$\text{and} \qquad T = W\frac{ac}{hl}.$$

341. *To find the pressure on the peg in the preceding problem.*

The pressure on the peg is equal and opposite to the resistance. Let R denote this force, then since the sides of the triangle ACB are proportional to the three forces acting upon the beam,

$$R : W :: AD : AC \text{ or } h.$$

But CD : CB :: a : l; therefore, substituting $\frac{l}{a}$ for n in the expression given in the foot note, page 212, we have

$$\mathrm{AD} = \frac{1}{l}\sqrt{al^3 + (l-a)(lh^2 - ac^2)},$$

$$\mathrm{R} = \frac{\mathrm{W}}{hl}\sqrt{al^3 + (l-a)(lh^2 - ac^2)}.$$

If the beam be of uniform thickness and density, $a = \frac{1}{2}l$, and the preceding result becomes

$$\mathrm{R} = \frac{\mathrm{W}}{2h}\sqrt{2(h^2 + l^2) - c^2}.$$

342. As an illustration of the mode of using the general equations of equilibrium, we shall apply them to the solution of the problem just considered.

The body is acted on by three forces, R the resistance of the peg, T the tension of the cord, and W the weight.

Let θ be the angle which the direction of R makes with the beam, and let A, B, and C be the angles of the triangle ABC.

Let A be the origin of co-ordinates, and AB the axis of x; then the resolved parts of the three forces, and the co-ordinates of their points of application, are exhibited in the following table:—

X	Y	x	y
$\mathrm{R}\cos\theta$	$\mathrm{R}\sin\theta$	0	0
$-\mathrm{T}\cos\mathrm{B}$	$\mathrm{T}\sin\mathrm{B}$	l	0
$-\mathrm{W}\cos\mathrm{A}$	$-\mathrm{W}\sin\mathrm{A}$	a	0

Substituting these values in the three equations of equilibrium,

$$\Sigma . \mathrm{X} = 0,$$

$$\Sigma . \mathrm{Y} = 0,$$

$$\Sigma . (\mathrm{Y}x - \mathrm{X}y) = 0;$$

and we have for the determination of the three unknown quantities, R, θ, and T, the equations

$$\mathrm{R}\cos\theta - \mathrm{T}\cos\mathrm{B} - \mathrm{W}\cos\mathrm{A} = 0. \qquad \text{(i.)}$$

$$\mathrm{R}\sin\theta + \mathrm{T}\sin\mathrm{B} - \mathrm{W}\sin\mathrm{A} = 0. \qquad \text{(ii.)}$$

$$\mathrm{T}l\sin\mathrm{B} - \mathrm{W}a\sin\mathrm{A} = 0. \qquad \text{(iii.)}$$

Hence, from equation iii.,

$$\mathrm{T} = \mathrm{W}\frac{a\sin\mathrm{A}}{l\sin\mathrm{B}},$$

$$= \mathrm{W}\frac{ac}{hl}.$$

From equations i. and ii., by division, we obtain

$$\tan\theta = \frac{\mathrm{W}\sin\mathrm{A} - \mathrm{T}\sin\mathrm{B}}{\mathrm{W}\cos\mathrm{A} + \mathrm{T}\cos\mathrm{B}};$$

$$= \frac{l - a}{l\cot\mathrm{A} + a\cot\mathrm{B}}.$$

Hence θ is found. Substituting, in equation ii., the value of $\sin\theta$, given by the equation just obtained, and we have

$$\mathrm{R} = \frac{\mathrm{W}\sin\mathrm{A}}{l}\sqrt{\{(l-a)^2 + (l\cot\mathrm{A} + a\cot\mathrm{B})^2\}}.$$

This result differs in form only from that obtained in Art. 341, for

$$(l-a)^2 + (l\cot\mathrm{A} + a\cot\mathrm{B})^2 = \frac{l^2}{\sin^2\mathrm{A}} + \frac{a^2}{\sin^2\mathrm{B}} + \frac{2al\cos(\mathrm{A}+\mathrm{B})}{\sin\mathrm{A}\sin\mathrm{B}};$$

$$= \frac{1}{\sin^2\mathrm{A}}\left\{l^2 + \frac{a^2\sin^2\mathrm{A}}{\sin^2\mathrm{B}} + 2al\frac{\sin\mathrm{A}}{\sin\mathrm{B}}.\cos(\mathrm{A}+\mathrm{B})\right\};$$

$$= \frac{1}{\sin^2\mathrm{A}}\left\{l^2 + \frac{a^2c^2}{h^2} + \frac{2acl}{h}\cos(\mathrm{A}+\mathrm{B})\right\};$$

$$= \frac{1}{h^2\sin^2\mathrm{A}}(h^2l^2 + a^2c^2 - 2achl\cos\mathrm{C}).$$

But $2ch\cos\mathrm{C} = c^2 + h^2 - l^2$; therefore

$$\mathrm{R} = \frac{\mathrm{W}}{hl}\sqrt{\{h^2l^2 + a^2c^2 - al(c^2 + h^2 - l^2)\}}.$$

343. *One extremity* A *of a beam* AB (fig. Art. 340) *rests against a vertical wall, and a cord tied to the other extremity* B *is fastened at* C, *a point in the wall above* A, *to find the position of the beam when* A *is on the point of sliding down the wall.*

Let l be the length of the beam, and $\frac{lm}{n}$ the length of the cord. Let G be the centre of gravity of the beam, and let AG : GB : : 1 : n, whence $AG = \frac{l}{n+1}$. Since the beam is in equilibrium, the line AD must be the direction of the resistance of the wall; and since A is on the point of sliding down the wall, the angle CAD must be the complement of the angle of repose. Let ϵ be the angle of repose, and let $\tan \epsilon = \mu$, then μ is the coefficient of friction. When the beam is in the position required, let $\theta =$ the angle CAB.

In the triangle ADG,

$$DG : AG :: \sin DAG : \sin ADG,$$

$$:: \sin (\theta + \epsilon - 90^\circ) : \sin (90^\circ - \epsilon),$$

$$:: -\cos (\theta + \epsilon) : \cos \epsilon;$$

$$\therefore \quad DG = -\frac{l}{n+1} \cdot \frac{\cos (\theta + \epsilon)}{\cos \epsilon};$$

and therefore, since AC : DG : : AB : GB : : $n+1 : n$,

$$AC = -\frac{l}{n} \cdot \frac{\cos (\theta + \epsilon)}{\cos \epsilon}, \quad \text{and } \tan \epsilon = \mu$$

$$= \frac{l}{n}(\mu \sin \theta - \cos\theta).$$

But $BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos \theta$, and hence

$$\frac{l^2 m^2}{n^2} = l^2 + \frac{l^2}{n^2}(\mu \sin \theta - \cos \theta)^2 - \frac{2l^2}{n}(\mu \sin \theta \cos \theta - \cos^2\theta), \quad \text{(a)}$$

Whence, multiplying by $\frac{n^2}{l^2}$, and writing $1-\cos^2\theta$ for $\sin^2\theta$, we have

$$m^2-n^2-\mu^2=\cos^2\theta\ (1+2n-\mu^2)-2\mu(1+n)\sin\theta\cos\theta,$$
$$=\cos^2\theta\{(1+2n-\mu^2)-2\mu(1+n)\tan\theta\}.$$

For $\cos^2\theta$ write $\frac{1}{1+\tan^2\theta}$, then

$$(m^2-n^2-\mu^2)\tan^2\theta+2\mu(1+n)\tan\theta+m^2-(1+n)^2=0.$$

Whence we obtain

$$\tan\theta=\frac{-\mu(1+n)\pm\{(m^2-n^2)(1+n)^2-m^2(m^2-n^2-\mu^2)\}^{\frac{1}{2}}}{m^2-n^2-\mu^2}.$$

344. Show that when the beam rests in a horizontal position, the length of the cord is equal to

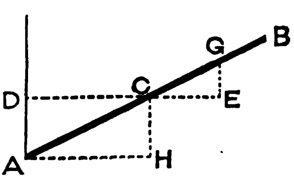
$$\frac{l}{n}\sqrt{(n^2+\mu^2)},$$ — — Eq (α).

and that when the beam is inclined at an angle of 45° with the horizontal line, the length of the cord is equal to

$$\frac{l}{n\sqrt{2}}\left\{n^2+(1+\mu+n)^2.\right\}^{\frac{1}{2}}$$

345. *A beam rests freely upon a peg, with one end against a smooth vertical wall; required the position of equilibrium, and the pressures on the wall and peg.*

Let AB be a beam resting upon the peg C and against the wall AD. Let G be the centre of gravity of the beam, and W its weight. Draw DE, AH, perpendicular to AD, and GE, CH, parallel to AD. Let $GA=a$, and $CD=b$. When there is equilibrium, let $CA=x$.



The resistance of the wall is equal and opposite to the pressure

of the beam against it. Let R denote this resistance, and Q the resistance of the peg. The beam then is acted on by three forces; by R acting perpendicularly to the wall, by Q acting perpendicularly to the beam, and by W acting vertically. The sides of the triangle ACD are severally perpendicular to the directions of these forces, and therefore, since there is equilibrium, Q : W : : AC : CD, and R : W : : AD : CD;

$$\therefore \qquad Q = W\frac{x}{b}, \qquad \text{(i.)}$$

$$\text{and} \qquad R = W\frac{\sqrt{(x^2 - b^2)}}{b} \qquad \text{(ii.)}$$

In order that equilibrium may subsist, the resultant of R and W must pass through C; therefore

$$R \times CH = W \times CE.$$

But $CH = AD = \sqrt{(x^2 - b^2)}$, and since the triangles CEG, CAD are similar,

$$CE : CG :: CD : AC,$$

$$\text{or,} \qquad CE : a - x :: b : x.$$

$$\text{Hence,} \qquad R\sqrt{(x^2 - b^2)} = Wb \, . \frac{a-x}{x} \qquad \text{(iii.)}$$

The equations i. ii. and iii. are sufficient to determine the three unknown quanities R, Q, and x. For substituting in equation iii. the value of R given by equation ii. we have

$$x^2 - b^2 = b^2 . \frac{a-x}{x};$$

$$\therefore \qquad x^3 = b^2 a, \text{ or } x = b\sqrt[3]{\left(\frac{a}{b}\right)}.$$

Substituting this value of x in equations i. and ii. we have

$$Q = W\sqrt[3]{\left(\frac{a}{b}\right)} \text{ and } R = W\left\{\left(\frac{a}{b}\right)^{\frac{2}{3}} - 1.\right\}^{\frac{1}{2}}$$

346. In the preceding example, if the beam, instead of resting freely upon the peg C, were fixed by a pivot at any point C, the only condition of equilibrium is that the resultant of R and W pass through C. Therefore, if $CA = c$, the pressure on the wall is obtained immediately from equation iii.; whence

$$R\sqrt{(c^2 - b^2)} = Wb\frac{a-c}{c}; \;\therefore\; R = W\frac{b}{c}\cdot\frac{a-c}{\sqrt{(c^2-b^2)}}.$$

347. *One extremity* B *of a beam* AB (fig. Art. 269) *rests against a vertical wall, and the other* A, *upon a horizontal plane; to find the position of the beam when on the point of slipping.*

Let l be the length of the beam, W its weight, G its centre of gravity, and let AG be equal to a. Let R be the normal resistance at A, and Q that at B. Let μ_1 be the coefficient of friction for the plane, and μ_2 for the wall; then the force of friction at A will be a horizontal force $\mu_1 R$ acting in the direction of P, and the force of friction at B will be a vertical force $\mu_2 Q$ acting upwards.

Let θ be the angle BAC when the beam is on the point of slipping. Then, resolving the forces into two sets at A and B, we have (X denoting the reaction of A and B),

$$\mu_1 R : R - W\frac{l-a}{l} :: \cos\theta : \sin\theta. \qquad \text{(i.)}$$

$$\mu_1 R : X :: \cos\theta : 1. \qquad \text{(ii.)}$$

$$Q : X :: \cos\theta : 1. \qquad \text{(iii.)}$$

$$Q : W\frac{a}{l} - \mu_2 Q :: \cos\theta : \sin\theta. \qquad \text{(iv.)}$$

From ii. and iii. it follows, that $Q = \mu_1 R$, whence from i. and iv.

$$R - W\frac{l-a}{l} = W\frac{a}{l} - \mu_1\mu_2 R;$$

$$\therefore \qquad R = \frac{W}{1 + \mu_1\mu_2}.$$

Substituting this value of R in i. we obtain

$$\frac{W\mu_1 \tan\theta}{1 + \mu_1\mu_2} = \frac{W}{1 + \mu_1\mu_2} - W\frac{l-a}{l};$$

$$\therefore \qquad \tan\theta = \frac{a - \mu_1\mu_2(l-a)}{l\mu_1}.$$

348. *One extremity of a beam is fastened by a pivot, and the other rests upon a smooth horizontal plane; required the upward vertical pressure upon the pivot when the foot of the beam is acted on by a given horizontal force, the weight of the beam itself being neglected.*

Let the beam AB (see fig. Art. 269) be fastened by a pivot at B, and let a force P be applied horizontally at A. Let the angle BAC be equal to α.

The beam is in equilibrium from the force P, the resistance of the plane acting vertically at the point A and the resistance of the pivot. But when three forces are in equilibrium, any one must be opposite to the resultant of the other two; hence, the resistance of the pivot will act along the line BA. Let R′ denote the resistance of the pivot, then

$$R' : P :: \sin 90^\circ : \sin(90^\circ - \alpha);$$

$$\therefore \qquad R' = \frac{P}{\cos\alpha}.$$

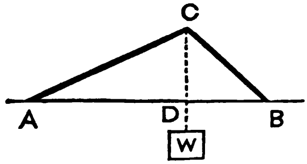
The vertical pressure required is the vertical component of R′;

$$\begin{aligned} \therefore \qquad \text{vertical pressure on } B &= R' \cos ABC; \\ &= R' \sin\alpha; \\ &= P\tan\alpha. \end{aligned}$$

349. *Let* AC, BC *be two beams, of uniform thickness and density, resting upon a horizontal plane at* A *and* B, *and fastened together at*

C by a pin; let A and B be connected by a cord: required the tension in the cord when a weight w *hangs from* C.

Let w_1 = the weight of the beam AC, and w_2 = the weight of the beam BC. Then, since the beams are of uniform thickness and density, we may resolve w_1 acting at the centre of gravity of AC into two forces, each equal to $\frac{1}{2}w_1$ acting vertically downwards at A and C; and, similarly, w_2 may be resolved into two forces, each equal to $\frac{1}{2}w_2$ acting vertically downwards at B and C. Hence, the whole force acting downwards at C is $w + \frac{1}{2}(w_1 + w_2)$. Let $W = w + \frac{1}{2}(w_1 + w_2)$. Since there is equilibrium, the upward vertical pressures upon C must equal W. Let T = the tension in the cord; let the angle CAB = α_1, and the angle CBA = α_2.



By the preceding article, the vertical thrust of the beam AC upon C is T tan α_2, and the vertical thrust of the beam BC is T tan α_1. Hence,

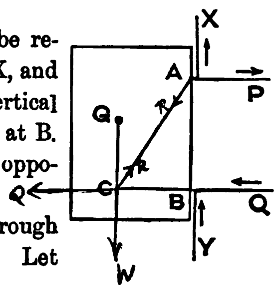
$$T \tan \alpha_1 + T \tan \alpha_2 = W;$$

$$\therefore \qquad T = \frac{W}{\tan \alpha_1 + \tan \alpha_2}.$$

350. *To find the horizontal strain upon the hinges of a door.*

Let G be the centre of gravity of the door, and W its weight. Let the hinges be at A and B.

Instead of the hinges, let the door be regarded as supported by a vertical force X, and a horizontal force P acting at A, and a vertical force Y and a horizontal force Q acting at B. It is evident that P and Q must act in opposite directions.



Let GC be the vertical line drawn through G. Draw BC perpendicular to GC. Let BC = a, and AB = b.

Remove the point of application of Y from B to A, of Q from B to C, and of W from G to C.

The resultant of W and Q at C must be equal and opposite to the resultant of P and X + Y at A. Each resultant must therefore act along AC. Let R represent the value of either; then

$$Q : W :: BC : AB,$$

or, $$Q : W :: a : b;$$

$\therefore$ $$Q = W\frac{a}{b},$$

also, $$Q : R :: BC : AC.$$

Again, since the forces at A are in equilibrium,

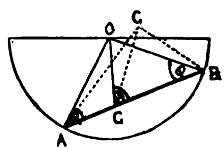
$$P : R :: BC : AC;$$

therefore, $$P = Q;$$

or the horizontal strain on each hinge is the same in amount, while opposite in direction.

351. *To find the position of equilibrium of a beam resting in a smooth hemispherical bowl.*

Let AB be the beam, and G its centre of gravity. Let $2l$ = the length of the beam, a = AG, and r = the radius of the bowl. The reactions of the bowl will act along the radii AO, BO. Consequently, in the position of equilibrium, the vertical line through O must pass through G. Let θ = the angle OGB, and let ϕ = the angle OAB = the angle OBA. Then



$$AG : AO :: \sin AOG : AGO,$$

or, $$a : r :: \sin(\theta - \phi) : \sin\theta$$

$\therefore$ $$\frac{\sin(\theta - \phi)}{\sin\theta} = \frac{a}{r};$$

$\therefore$ $$\cos\phi - \cot\theta \sin\phi = \frac{a}{r};$$

$\therefore$ $$\cot\theta = \frac{r\cos\phi - a}{r\sin\phi}.$$

But $r \cos \phi = l$, and therefore $r \sin \phi = \sqrt{(r^2 - l^2)}$;

$$\therefore \qquad \cot \theta = \frac{l - a}{\sqrt{(r^2 - l^2)}}.$$

352. *To find the position of a beam resting in a hemispherical bowl when on the point of slipping.*

As in the preceding, let θ be the angle made by the beam with the vertical line through its centre of gravity, and ϕ the angle between the beam and the radius through either extremity. Let ϵ be the angle of repose.

Let AB (fig. Art. 351) be the beam in the position required, and let the reactions of the bowl act in the directions AC, BC. Then the angle CAB $= \phi - \epsilon$, and the angle CBA $= \phi + \epsilon$. As before, the vertical line through C must pass through G. Let CG be vertical; then, by hypothesis, the angle GCB $= \theta$.

$$\text{AG} : \text{GC} :: \sin \text{ACG} : \sin \text{CAG},$$

or, $$a : \text{GC} :: \sin(\theta - \phi + \epsilon) : \sin(\phi - \epsilon).$$

And $$\text{GC} : \text{GB} :: \sin \text{CBG} : \sin \text{GCB},$$

or, $$\text{GC} : 2l - a :: \sin(\phi + \epsilon) : \sin(\theta + \phi + \epsilon).$$

Hence, $$\frac{\sin(\theta - \phi + \epsilon)}{\sin(\theta + \phi + \epsilon)} \cdot \frac{\sin(\phi + \epsilon)}{\sin(\phi - \epsilon)} = \frac{a}{2l - a}.$$

$$\therefore \quad \frac{\sin \theta \cos(\phi - \epsilon) - \cos \theta \sin(\phi - \epsilon)}{\sin \theta \cos(\phi + \epsilon) + \cos \theta \sin(\phi + \epsilon)} \cdot \frac{\sin(\phi + \epsilon)}{\sin(\phi - \epsilon)} = \frac{a}{2l - a}.$$

Dividing numerator and denominator by $\sin \theta \sin(\phi - \epsilon) \sin(\phi + \epsilon)$,

$$\frac{\cot(\phi - \epsilon) - \cot \theta}{\cot(\phi + \epsilon) + \cot \theta} = \frac{a}{2l - a}.$$

Whence $\cot \theta = \frac{1}{2l}\{(2l - a)\cot(\phi - \epsilon) - a \cot(\phi + \epsilon)\}$.

The value of ϕ is known from the equation $r \cos \phi = l$, and ϵ is given, therefore θ is determined.

353. *A beam, leaning against a small cylindrical peg* B, *rests with one end* A *upon a horizontal plane; the coefficient of friction is the same between the beam and the peg, and between the beam and the plane: determine the position of the beam when on the point of slipping.*

Let a be the distance of the centre of gravity of the beam from A, θ the inclination of the beam to the horizontal plane, and b the height of the peg above the plane.

Let R_1 be the resistance of the plane, and R_2 the resistance of the peg, and ϵ the angle of repose; then,

$$\begin{aligned} R_1 \hat{} R_2 &= \theta, \\ R_2 \hat{} W &= 180^\circ - (\theta - \epsilon), \\ W \hat{} R_1 &= 180^\circ - \epsilon. \end{aligned}$$

Hence, by the triangle of forces,

$$R_1 = \frac{W \sin(\theta - \epsilon)}{\sin\theta},$$

and

$$R_2 = \frac{W \sin\epsilon}{\sin\theta},$$

Also, taking the moments of R_2 and W about A, we have

$$\frac{b \sin\epsilon \cdot \cos\epsilon}{\sin^2\theta} = a \cos\theta;$$

and therefore

$$\sin^2\theta \cos\theta = \frac{b \sin\epsilon \cos\epsilon}{a},$$

or,

$$\cos^3\theta - \cos\theta + \frac{b \sin 2\epsilon}{2a} = 0.$$

354. *A bar* AB, *whose weight is* W, *rests with one end* A *upon a horizontal plane, a cord fastened to* B *passes over a fixed pulley placed above the plane and carries a weight* P; *determine the position of equilibrium, and the pressure upon the plane, when* A *is on the point of slipping towards the pulley, disregarding the friction of the pulley.*

Let θ be the inclination of the beam to the horizontal line, and ϕ that of the cord. Let R be the resistance of the plane, and ϵ the angle of repose; then,

$$\hat{RW} = 180° - \epsilon,$$

$$\hat{WP} = 90° + \phi,$$

$$\hat{PR} = 90° - (\phi - \epsilon).$$

Therefore, by the triangle of forces,

$$\frac{W}{P} = \frac{\cos(\phi - \epsilon)}{\sin \epsilon}. \qquad \text{(i.)}$$

$$\frac{R}{P} = \frac{\cos \phi}{\sin \epsilon}. \qquad \text{(ii.)}$$

Let l be the length of the bar, and a the distance of its centre of gravity from A; then, taking the moments about A,

$$Wa \cos \theta = Pl \sin(\phi - \theta);$$

and therefore

$$\tan \theta = \frac{Pl \sin \phi - Wa}{Pl \cos \phi}. \qquad \text{(iii.)}$$

The value of ϕ is determined by equation i., and thence the value of R is found from equation ii., and the value of θ from equation iii.

EXAMPLES.

1. Two weights, P and W, balance each other on a straight lever moveable about a cylindrical pivot, find the ratio of P to W when P is on the point of descending; the arms of the lever being a and b respectively, r the radius of the pivot, and μ the coefficient of friction.

The ratio of P to W is equal to $\frac{b + \mu r}{a + \mu r}$.

2. If a beam rest with one extremity A in a smooth hemispherical bowl, and with the other projecting beyond the edge, show that in the position of equilibrium,

$$\sin\theta = \frac{a + \sqrt{(a^2 + 32r^2)}}{8r},$$

θ being the inclination of the beam to the vertical line, r the radius of the bowl, and a the distance of A from the centre of gravity of the beam.

3. What must be the length of a beam in order that, when resting within a sphere, its limiting position may be vertical?

Let μ be the coefficient of friction, and r the radius of the sphere, then the required length is equal to

$$\frac{2r}{\sqrt{(1 + \mu^2)}}.$$

4. Find the position of equilibrium of a balance when unequal weights P, Q are suspended from A and B, the extremities of the beam. Let $AB = 2a$, W the weight of the balance, b the distance of AB below the fulcrum, and h the distance of the centre of gravity of the balance below the fulcrum. Then, if θ be the angle the beam makes with the horizontal line when there is equilibrium,

$$\tan\theta = \frac{(P - Q)a}{(P + Q)b + Wh}.$$

5. One extremity A of a beam rests against a smooth hemispherical bowl, and the other against a smooth vertical plane passing through the centre of the bowl—the radius of the bowl being greater than the length of the beam—then, if θ be the inclination of the beam to the vertical plane when in equilibrium, l the length of the beam, a the distance of its centre of gravity from A, and r the radius of the bowl,

$$\sin\theta = \sqrt{\left(\frac{l^4 - a^2r^2}{l^4 - a^2l^2}\right)}.$$

6. The lower extremity A of a beam rests against a vertical wall, a cord attached to the beam at a point B, between the centre of gravity and A, is fastened to the wall at a point C vertically above A, show that if ϕ be the inclination of the cord to the wall, and θ that of the beam, the position of the beam when A is on the point of slipping down is determined by the following equations:—

$$a(\cot\phi+\mu)=b(\cot\theta-\mu),$$
$$c\sin\phi=b\sin\theta,$$

a being the distance of B from the centre of gravity, and b the distance of B from A.

7. In the preceding, if the wall be perfectly smooth, and $b=a$, then will $c=a$, and the beam will rest in any position so long as A is below C.

8. A beam AB rests upon two inclined planes whose inclinations are α_1 and α_2 respectively, show that if a be the distance of the centre of gravity from A, b its distance from B, ϵ the angle of repose, and θ the inclination of the beam to the vertical line when A is on the point of slipping up,

$$\tan\theta=\frac{a+b}{b\cot(\alpha_2-\epsilon)-a\cot(\alpha_1+\epsilon)}.$$

9. A thin hemispherical bowl is mounted with a cylindrical rim formed of a substance whose specific gravity is four times that of the substance of the bowl, what must be the height of the rim that the bowl may rest upon a horizontal plane in any position?

The height of the rim is equal to half the radius of the bowl.

10. In the preceding, if the bowl be thick, having its external and internal radii equal to a and b respectively, show that the height of the rim is equal to

$$\left(\frac{a^2+b^2}{8}\right)^{\frac{1}{2}}.$$

11. Four balls, of equal radii but unequal weights, rest within a hollow sphere in such a way that their centres lie in the same vertical plane, and each ball is in contact with the sphere, find the position of equilibrium.

Let W_1, W_2, W_3, W_4, be the weights of the balls. Let θ be the inclination to the horizon of the line joining the centre of the first ball with the centre of the sphere, and let a be the angle subtended at the centre of the sphere by the line joining the centres of any two adjacent balls, then

$$\tan\theta = \frac{W_1 + W_2 \cos a + W_3 \cos 2a + W_4 \cos 3a}{W_2 \sin a + W_3 \sin 2a + W_4 \sin 3a}.$$

DYNAMICS.

CHAPTER VII.

ON THE IMPACT OF BODIES.

355. When any two bodies come into collision, the internal disposition of their particles is disturbed to a greater or less extent. It is found by experience, that when any such disturbance takes place, all matter possesses the power of returning in some measure to its original state. To this property the name *elasticity* is given. Its effect when one body impinges upon another is to cause them to separate, and the magnitude of the force is measured by the ratio which the velocity of the recoil bears to the velocity of the approach. The value of this ratio is termed the measure or modulus of the elasticity. When this ratio is unity—that is, when the velocity of the recoil equals the velocity of the approach—the elasticity is said to be *perfect;* in all other cases it is called *imperfect.*

356. By the impact of one body upon another, a change is produced in the motion of both. These changes, though seeming to be effected instantaneously, are not so in reality. That a finite though inappreciably small period of time is required for their production is shown by experiments of the following kind. Let

two ivory balls, one of which is stained with ink, impinge gently upon each other, a mere speck of ink will be found upon the unstained ball. Let the balls impinge with a greater force, and the speck becomes enlarged into a small circular area; and the greater the force the larger is this spot. As two spheres can touch each other in one point only, the experiment shows that during the impact the form of the balls underwent a change, so as to allow certain portions of the surfaces to come into contact, and that the centres of the balls approached nearer to each other than when the contact commenced. Motion, therefore, has taken place over a finite distance; and as there is no reason for supposing the velocity has been otherwise than finite, it follows that the time of the motion is finite also.

357. DEFINITIONS. The product of the mass of any body and its velocity is called its *momentum*.* momentum $= Mv$.

* Since the masses of bodies vary both as their volume and as their density, if V, V′ represent the volumes, ρ, ρ' the densities, and M, M′ the masses, respectively,

$$M : M' :: V\rho : V'\rho',$$

or,

$$\frac{M}{M'} = \frac{V}{V'} \cdot \frac{\rho}{\rho'};$$

whence, if $M' = 1$, $V' = 1$, and $\rho' = 1$, we have

$$M = V\rho;$$

the meaning of which is, that the mass of any body contains the unit of mass as many times as the product of the number of times the volume contains the unit of volume, and the number of times the density contains the unit of density.

Also, by the third law of motion,

$$Mf : M'f' :: P : P',$$

or,

$$\frac{P}{P'} = \frac{M}{M'} \cdot \frac{f}{f'};$$

whence, if $P' = 1$, $M' = 1$, $f' = 1$, we obtain the equation

$$P = Mf;$$

the meaning of which is, that any pressure P contains the unit of pressure as many times as the product of the number of times the mass contains the unit of mass, and the number of times the force contains the unit of force.

The product of the mass of any body and the square of its velocity is called its *vis viva*. vis viva = Mv^2.

358. *When equal pressures produce motion in different bodies, the momenta generated in each are equal.*

Let a pressure P produce motion in bodies whose masses are M and M′, and let V, V′ be the velocities at the expiration of any time t.

First; Let the pressures be constant during the time t, then the velocities generated are uniformly accelerated, and therefore the forces are $\frac{V}{t}$ and $\frac{V'}{t}$; and consequently, by the third law of motion,

$$\frac{MV}{t} : \frac{M'V'}{t} :: P : P;$$

and therefore $$MV = M'V'.$$

Secondly; Let the pressures, while always equal to each other, be variable during the time t. Divide the interval t into n equal periods, n being an indefinitely large number. During any one of these periods the pressures may be regarded as constant; and if v_1, v_2, &c. v'_1, v'_2, &c. denote the velocities generated during

Substituting in this equation the value of M obtained above, we have

$$P = V\rho f.$$

The unit of force is, as already stated in Part I., the force which generates a velocity of one foot in one second; the unit of volume is the cubic foot; and the unit of density is the density of distilled water. When V, ρ, and f are each unity, P=1, or the unit of pressure is the pressure exerted by one foot of distilled water under the action of a force which generates a velocity of one foot in one second.

When gravity is the force acting upon a body, the pressure is called weight, and hence

$$W = V\rho g,$$
$$= 32{\cdot}2V\rho.$$

each successive period, it follows, from what has been established above, that

$$Mv_1 = M'v'_1,$$

$$Mv_2 = M'v'_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Mv_n = M'v'_n.$$

Adding these together, we have

$$M(v_1 + v_2 + \ldots\ldots + v_n) = M'(v'_1 + v'_2 + \ldots\ldots + v'_n).$$

But $v_1 + v_2 + \ldots\ldots + v_n = V$, and $v'_1 + v'_2 + \ldots\ldots + v'_n = V'$; therefore

$$MV = M'V'.$$

If the pressures producing motion be the mutual action of two bodies in contact, the pressures are equal in magnitude but opposite in direction; and hence in such cases the momenta generated are equal and opposite.

259. *Two smooth spherical and inelastic bodies impinge directly upon each other, to determine their common velocity immediately after impact.*

Let M and M′ be the masses of the bodies, V, V′ their velocities at the instant of impact, and v their common velocity immediately after impact. Since the duration of the impact is indefinitely short, the velocities of the bodies at the instant of impact may be taken for the velocities they would have had at the moment when the impact ceases, and consequently the difference between the velocity after and before impact may be taken for the change of velocity produced, or the velocity generated by impact.

Let the positive direction of motion be that of the impinging body. Let both the bodies be moving in the positive direction.

Then, since M impinges upon M′, V must be greater than V′. Hence,

$$v - V = \text{velocity generated in M by impact,}$$
$$v - V' = \quad " \quad " \quad M' \quad "$$

Therefore, by the preceding Article,

$$(v - V)\, M = -(v - V')\, M';$$
$$\therefore \qquad v = \frac{MV + M'V'}{M + M'}. \qquad \text{(i.)}$$

If the bodies be moving in opposite directions, V′ will be negative, or

$$v = \frac{MV - M'V'}{M + M'}, \qquad \text{(ii.)}$$

If M impinge upon M′ at rest, $V' = 0$. Therefore,

$$v = \frac{MV}{M + M'}. \qquad \text{(iii.)}$$

360. From equations i. and ii. the value of the velocities generated by impact in the two bodies may be determined. For

$$\text{velocity generated in } M = v - V,$$
$$= -\frac{M'(V \mp V')}{M + M'},$$

$$\text{and velocity generated in } M' = v - V',$$
$$= \frac{M\,(V \mp V')}{M + M'},$$

the upper signs being taken when the bodies move in the same direction, and the lower when they move in opposite directions. It will be seen that the velocity generated by impact in the body moving with the greater velocity is negative, but in the other body is positive.

361. From equation ii. (Art. 359) it follows, that when two bodies moving in opposite directions impinge upon each other, there will be rest after impact, if

$$MV = M'V'.$$

362. *Two smooth spherical and inelastic bodies impinge obliquely upon each other, to determine their velocities and the directions of their motions after impact.*

Let M and M′ be the two bodies, moving with velocities V, V′ in directions making with the common tangent at the point of contact the angles α, α'. The mutual pressures during impact act along the common normal. The velocities V, V′ may be resolved into velocities $V \cos \alpha$, $V' \cos \alpha'$ acting along the common tangent, and $V \sin \alpha$, $V' \sin \alpha'$ acting along the common normal. The former, since the bodies are smooth, will not be affected by the impact; the latter will each become, according to Art. 359,

$$\frac{MV \sin \alpha + M'V' \sin \alpha'}{M + M'}.$$

Hence, if v and v' be the velocities of the bodies after impact, acting in directions making with the common tangent the angles β, β',

$$v \cos \beta = V \cos \alpha,$$

$$v \sin \beta = \frac{MV \sin \alpha + M'V' \sin \alpha'}{M + M'},$$

Whence $$v^2 = V^2 \cos^2 \alpha + \left(\frac{MV \sin \alpha + M'V' \sin \alpha'}{M + M'}\right)^2,$$

and $$\tan \beta = \frac{MV \sin \alpha + M'V' \sin \alpha'}{V \cos \alpha \,(M + M')}.$$

Similarly, $$v'^2 = V'^2 \cos^2\alpha' + \left(\frac{MV \sin\alpha + M'V' \sin\alpha'}{M+M'}\right)^2,$$

and $$\tan\beta' = \frac{MV \sin\alpha + M'V' \sin\alpha'}{V' \cos\alpha' (M+M')}.$$

363. Hence, if $V \cos\alpha = V' \cos\alpha'$, then $\beta = \beta'$ and $v = v'$, or the bodies after impact will both move in the same direction and with a common velocity.

364. Again, if the directions of V and V′ fall upon different sides of the common tangent, α' will be negative, and therefore

$$\tan\beta = \frac{MV \sin\alpha - M'V' \sin\alpha'}{V \cos\alpha (M+M')},$$

and $$\tan\beta' = \frac{MV \sin\alpha - M'V' \sin\alpha'}{V' \cos\alpha' (M+M')}.$$

Consequently, if $MV \sin\alpha = M'V' \sin\alpha'$, then $\beta = 0$, and $\beta' = 0$; or the two bodies after impact will both move in the direction of the common tangent, and their velocities will be $V \cos\alpha$ and $V' \cos\alpha'$ respectively.

365. Hence, also, if M impinge obliquely upon M′ at rest, making $V' = 0$,

$$v = V\left\{\cos^2\alpha + \frac{M^2 \sin^2\alpha}{(M+M')^2}\right\}^{\frac{1}{2}},$$ ∴ when $M' = \infty$, $v = V\cos\alpha$

$$\tan\beta = \frac{M \tan\alpha}{M+M'},$$ ∴ when $M' = \infty$, $\tan\beta = 0$

$$v' = \frac{MV \sin\alpha}{M+M'},$$

$$\tan\beta' = \infty; \qquad \text{or } \beta' = 90°.$$

366. In the last, let $M' = \infty$, and we have the case of a body impinging upon an immoveable surface. Dividing numerator and denominator of the values of v and $\tan\beta$ by M',

$$v = \frac{V}{\frac{M}{M'} + 1} \cdot \sqrt{\left(\frac{M^2}{M'^2} + 2\frac{M}{M'}\cos\alpha + \cos^2\alpha\right)},$$

and
$$\tan\beta = \frac{\frac{M}{M'}\tan\alpha}{\frac{M}{M'} + 1}.$$

But if $M' = \infty$, $\frac{M}{M'} = 0$; therefore

$$v = V\cos\alpha,$$
$$\tan\beta = 0,$$

or the body moves along the common tangent with a velocity $V\cos\alpha$; and hence, if the fixed surface be a plane, a body impinging obliquely upon it will, after impact, move along the plane.

367. *Two imperfectly elastic spherical bodies impinge directly upon each other, to find their velocities immediately after impact.*

Let the two bodies move in the same direction with velocities V, V', and let M, M' be their masses. Let e be the modulus of elasticity, and v, v' the velocities of the bodies immediately after impact. Then the velocity of approach is $V - V'$, and the velocity of recoil $v' - v$; wherefore

$$\frac{v' - v}{V - V'} = e,$$

and by Art. 358,

$$(v - V)M = -(v' - V')M';$$

$$\therefore \quad v = \frac{MV + M'V' - eM'(V - V')}{M + M'},$$

and
$$v' = \frac{MV + M'V' + eM(V - V')}{M + M'}.$$

$$Mv + M'v' = \frac{M^2V + MM'V' + MM'V + M'^2V'}{M + M'} = MV + M'V'$$

∴ the whole momentum is unchanged by the impact.

and also $\frac{Mv + M'v'}{M + M'} = \frac{MV + M'V'}{M + M'}$.

If the two bodies move in contrary directions, let V′ be negative; then,

$$v = \frac{MV - M'V' - eM' (V + V')}{M + M'},$$

and $$v' = \frac{MV - M'V' + eM (V + V')}{M + M'}.$$

368. If, in the preceding, M impinge upon M′ at rest, $V' = 0$; and therefore

$$v = \frac{V (M - eM')}{M + M'},$$

$$v' = \frac{MV (1 + e)}{M + M'}.$$

If $M = eM'$, then $v = 0$, or the impinging body is at rest immediately after impact, and $v' = eV$, or M′ moves with the velocity eV.

369. In the expressions obtained in Art. 367 for the velocities after impact of two elastic bodies moving in the same or contrary directions, make $M' = M$, and $e = 1$, then

$$v = \frac{V \pm V' - (V \mp V')}{2} = \pm V',$$

$$v' = \frac{V \pm V' + (V \mp V')}{2} = V.$$

Hence, if two perfectly elastic bodies of equal masses, moving in the same or contrary directions with any velocities, impinge upon each other, they mutually exchange velocities.

370. *Two imperfectly elastic spherical bodies impinge obliquely upon each other, to find their velocities and the directions of their motions immediately after impact.*

Let M, M′ be the two bodies, moving with velocities V, V′, in directions making with the common tangent, at the point of contact, the angles α, α'. Let v, v' be the velocities after impact, and θ, θ' the angles which their directions make with the common tangent.

As in Art. 362, V and V′ may be resolved into velocities $V\cos\alpha$, $V'\cos\alpha'$, acting along the tangent; and $V\sin\alpha$, $V'\sin\alpha'$, acting along the normal. The former are unaltered by the impact; the latter will become respectively (Art. 367),

$$\frac{MV\sin\alpha + M'V'\sin\alpha' - eM'(V\sin\alpha - V'\sin\alpha')}{M+M'},$$

and $$\frac{MV\sin\alpha + M'V'\sin\alpha' + eM(V\sin\alpha - V'\sin\alpha')}{M+M'}.$$

Hence,

$$v\cos\theta = V\cos\alpha,$$

$$v\sin\theta = \frac{MV\sin\alpha + M'V'\sin\alpha' - eM'(V\sin\alpha - V'\sin\alpha')}{M+M'},$$

$$= \frac{V\sin\alpha(M - eM') + M'V'\sin\alpha'(1+e)}{M+M'};$$

$$\therefore\ v^2 = V^2\cos^2\alpha + \left\{\frac{V\sin\alpha(M-eM') + M'V'\sin\alpha'(1+e)}{M+M'}\right\}^2$$

$$\tan\theta = \frac{V\sin\alpha(M-eM') + M'V'\sin\alpha'(1+e)}{V\cos\alpha(M+M')}.$$

Similarly,

$$v'^2 = V'^2\cos^2\alpha' + \left\{\frac{V'\sin\alpha'(M'-eM) + MV\sin\alpha(1+e)}{M+M'}\right\}^2,$$

$$\tan\theta' = \frac{V'\sin\alpha'(M'-eM) + MV\sin\alpha(1+e)}{V'\cos\alpha'(M+M')}.$$

371. If M impinge obliquely upon M′ at rest, $V' = 0$;

$$\therefore \quad v^2 = V^2 \cos^2 \alpha + \left\{\frac{V \sin \alpha (M - eM')}{M + M'}\right\}^2,$$

$$\tan \theta = \tan \alpha \, . \, \frac{M - eM'}{M + M'},$$

$$v' = \frac{MV \sin \alpha (1 + e)}{M + M'},$$

$$\tan \theta' = \infty \; ; \text{ or } \theta' = 90°.$$

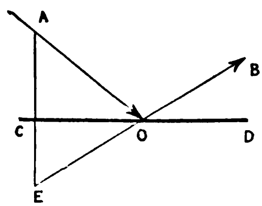
372. *An imperfectly elastic spherical body impinges obliquely upon an immoveable surface, to determine the velocity and the direction of the motion immediately after impact.*

In the preceding, let M′ be infinitely large in comparison with M, then $\frac{M}{M'}$ is indefinitely small.

Hence, $$v^2 = V^2 (\cos^2 \alpha + e^2 \sin^2 \alpha),$$
$$\tan \theta = -e \tan \alpha.$$

Let AO be the direction of the motion of the impinging body, O the point at which it meets the surface, and CD the common tangent. From A, any point in AO, draw ACE perpendicular to CD, and take $CE = eAC$;

$$\tan BOD = \tan COE,$$
$$= \frac{CE}{CO},$$
$$= -\frac{eAC}{CO}$$
$$= -e \tan AOC;$$



therefore OB is the direction of the motion after impact.

373. *To determine the motion of the centre of gravity of two bodies moving in the same straight line with uniform velocities.*

Let M, M′ be the masses of the two bodies, and V, V′ their

velocities. Let a, a' be the distances of their centres of gravity from any point in the line of their motion, and h the distance of their common centre of gravity from the same point; then (Art. 69),

$$h = \frac{Ma + M'a'}{M + M'}.$$

Let u be the velocity of the common centre of gravity, then at the expiration of any time t, a, a', and h will become $a + Vt$, $a' + V't$, $h + ut$ respectively. Therefore, as before,

$$h + ut = \frac{M(a + Vt) + M'(a' + V't)}{M + M'}.$$

Subtracting the former from this we obtain

$$u = \frac{MV + M'V'}{M + M'}.$$

374. *If two spherical bodies impinge directly, the velocity of their centre of gravity immediately after impact is the same as immediately before impact.*

Let M, M′ be the bodies, and V, V′ their velocities of impact. For the instant immediately preceding these velocities may be considered uniform; and therefore, by the preceding, if u denote the velocity of the centre of gravity,

$$u = \frac{MV + M'V'}{M + M'}.$$

If the bodies be inelastic, it has been already shown that their common velocity, and therefore the velocity of their centre of gravity, immediately after impact, is

$$\frac{MV + M'V'}{M + M'}.$$

The proposition is therefore true of inelastic bodies. If the bodies be elastic, and let v, v' be their velocities immediately after

impact, these for the instant may be regarded as uniform; and therefore if u' denote the velocity of the centre of gravity,

$$u' = \frac{Mv + M'v'}{M + M'}, \quad = \text{by note on p. 300} \; \frac{MV + M'V'}{M + M'} = u$$

Substituting the values of v and v', from Art. 367,

$$u' = \frac{M\{MV + M'V' - eM'(V - V')\} + M'\{MV + M'V' + eM(V - V')\}}{(M + M')^2},$$

$$= \frac{(MV + M'V')(M + M')}{(M + M')^2},$$

$$= \frac{MV + M'V'}{M + M'},$$

$$= u.$$

Since this Art. is true for direct impact, it can be shown to be true for oblique impact also. Vid. Parkinson, p. 203

375. *In the direct impact of imperfectly elastic bodies the sum of the* vires vivæ *after impact is less than before impact.*

By Art. 367,

$$v = \frac{MV + M'V' - eM'(V - V')}{M + M'},$$

$$v' = \frac{MV + M'V' + eM(V - V')}{M + M'};$$

$$\therefore \quad Mv^2 + M'v'^2 = \frac{(MV + M'V')^2 + e^2MM'(V - V')^2}{M + M'}.$$

In the numerator, add and subtract $MM'(V^2 + V'^2)$. Then

$$Mv^2 + M'v'^2 = \frac{(MV^2 + M'V'^2)(M + M') - (1 - e^2)MM'(V - V')^2}{M + M'},$$

$$= MV^2 + M'V'^2 - \frac{(1 - e^2)MM'(V - V')^2}{M + M'}.$$

Cor. If the bodies be perfectly elastic, $e = 1$, then

$$Mv^2 + M'v'^2 = MV^2 + M'V'^2,$$

or the sum of the *vires vivæ* is the same after impact as before.

X

Part. 1. Sec 53.

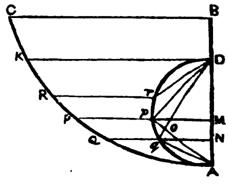
CHAPTER VIII.

ON CONSTRAINED MOTION.

376. *If a body fall down a smooth curve under the action of gravity, its velocity at any point is that due to the vertical height through which it has fallen.*

It has already been shown, that the velocity acquired by any body in falling down a smooth inclined plane is that due to the vertical height through which it has fallen. Hence, if a body fall in succession down a series of inclined planes, the velocity acquired will be that due to the total height through which it has fallen, provided that there be no change of velocity in passing from one plane to another. A smooth curve may be regarded as made up of a series of indefinitely small inclined planes, whose reaction, being always at right angles to the direction of the motion, produces no change in the velocity of a body moving over them. Consequently, the velocity at any point is that due to the vertical height through which the body has fallen.

377. *To find the time in which a body falls down an arc of a cycloid whose axis is vertical.*



Let the body fall from K down KA, the arc of a cycloid whose axis AB is vertical. Let P be any point in KA, and let PQ be the distance described in an indefinitely small portion of time. Draw the horizontal lines KD, PM, QN. On AD describe the semi-circle $AqpD$, cutting PM and QN in p and q. Draw

the chords Dp, Dq, Ap, Aq. The velocity of the body at P is, by the preceding Article $= \sqrt{(2g\text{DM})}$, and since PQ is indefinitely small, the velocity in it may be considered uniform; therefore, if t' be the time of describing PQ,

$$t' = \frac{\text{PQ}}{\sqrt{(2g\text{DM})}}.$$

It is shown in works that discuss the properties of the cycloid, that if AP be any arc of a cycloid, measured from the lowest point; then $\text{AP}^2 = 8a\,.\,\text{AM}$, where a is the radius of the generating circle, or the semi-axis of the cycloid. Let $\text{AB} = 2a$, then $\text{PQ} = \text{AP} - \text{AQ} = \sqrt{(8a\text{AM})} - \sqrt{(8a\text{AN})}$.

Whence,

$$t' = \sqrt{\left(\frac{4a}{g}\right)} \cdot \frac{\sqrt{(\text{AM})} - \sqrt{(\text{AN})}}{\sqrt{(\text{DM})}},$$

$$= \sqrt{\left(\frac{4a}{g}\right)} \cdot \frac{\sqrt{(\text{AM}\,.\,\text{AD})} - \sqrt{(\text{AN}\,.\,\text{AD})}}{\sqrt{(\text{DM}\,.\,\text{AD})}},$$

$$= \sqrt{\left(\frac{4a}{g}\right)} \cdot \frac{\text{A}p - \text{A}q}{\text{D}p}.$$

But since Q is indefinitely near to P, AO may be regarded as equal to Aq, and therefore $\text{A}p - \text{A}q = op$, and

$$t' = \sqrt{\left(\frac{4a}{g}\right)} \cdot \frac{op}{\text{D}p},$$

$$= \sqrt{\left(\frac{4a}{g}\right)} \times \text{angle } p\text{D}q.$$

Similarly,

$$\text{time of describing RP} = \sqrt{\left(\frac{4a}{g}\right)} \times \text{angle } r\text{D}p.$$

Hence, if the arc KA be divided into indefinitely small arcs, as KR, RP, &c., then, since the time of describing the whole arc is

equal to the sum of the times of describing the several parts, the time of describing

$$KA = \sqrt{\left(\frac{4a}{g}\right)} \times \text{the sum of the angles } KDr,\ rDP, \ \&c.$$

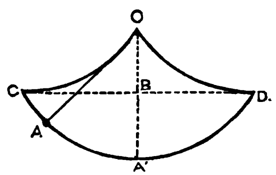
But the sum of these angles is $\frac{\pi}{2}$.

Therefore,

$$\text{time of describing } KA = \pi\sqrt{\left(\frac{a}{g}\right)}.$$

As this result is independent of the length of the arc KA, it shows that the time of falling to the lowest point of a cycloid down any arc is the same in all. From this property, the cycloid is called a *tautochronous* curve.

378. If OC, OD be two equal semi-cycloids, whose axis is $2a$, and a cord OA, equal in length to OC or OD, be fastened at O, and wrapped round OC or OD, the point A will, as the cord is unwrapped from the one semi-cycloid and wrapped around the other, describe a cycloid CAD, whose axis also is $2a$. Hence, if OC and OD be placed in a vertical plane, and a cord whose length is $4a$ be fastened at O, a body suspended at A will oscillate in cycloidal arcs, and consequently form a pendulum whose oscillations are equal in time, whatever their extent. If l be the length of such a pendulum, and t the time of an oscillation,



$$t = 2\pi\sqrt{\left(\frac{l}{4g}\right)},$$

$$= \pi\sqrt{\left(\frac{l}{g}\right)}.$$

If A move over a very small arc on either side of A′, the arc described does not sensibly differ from the arc of a circle; and

hence, if a pendulum oscillate through very small arcs of a circle, the times of oscillation will be the same, and will equal

$$\pi\sqrt{\left(\frac{l}{g}\right)}.$$

379. It will be seen from the foregoing result, that when the force of gravity remains the same, the time of the oscillation of a pendulum varies directly as the square root of the length, and that when the length is the same, the time varies inversely as the square root of the force of gravity.

380. *To find the number of oscillations gained or lost in a given period, when the length of a pendulum is slightly altered.*

Let l be the length of a pendulum making n oscillations in the time T. Let the length of the pendulum become $l+x$ (x being a very small quantity), and let n' be the number of oscillations then made in the time T. By Art. 378,

$$\frac{\mathrm{T}}{n} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}},$$

and $$\frac{\mathrm{T}}{n'} = \pi\left(\frac{l+x}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \frac{n'}{n} = \left(\frac{l}{l+x}\right)^{\frac{1}{2}} = \left(1+\frac{x}{l}\right)^{-\frac{1}{2}};$$

$$\therefore \quad \frac{n-n'}{n} = 1 - \left(1+\frac{x}{l}\right)^{-\frac{1}{2}},$$

$$= \frac{x}{2l} \text{ nearly, since } \frac{x}{l} \text{ is small};$$

$$\therefore \quad n - n' = \frac{nx}{2l}.$$

381. *To find the number of oscillations gained or lost in a given period, when the force of gravity is slightly altered.*

Let a pendulum, whose length is l, make n oscillations in a time T, when the force of gravity is g, and n' when the force of gravity is $g+x$. Then,

$$\frac{\mathrm{T}}{n} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}, \text{and } \frac{\mathrm{T}}{n'} = \pi\left(\frac{l}{g+x}\right)^{\frac{1}{2}};$$

$$\therefore \qquad \frac{n'}{n} = \left(1+\frac{x}{g}\right)^{\frac{1}{2}};$$

$$\therefore \qquad \frac{n'-n}{n} = \left(1+\frac{x}{g}\right)^{\frac{1}{2}} - 1,$$

$$= \frac{x}{2g} \text{ nearly};$$

$$\therefore \qquad n'-n = \frac{nx}{2g}.$$

382. *To find the height of a station above the earth by observing the number of oscillations lost by a pendulum in a given period, the earth being supposed to be spherical.*

Let a pendulum, whose length is l, make n oscillations in the time T, under the action of gravity, at the surface of the earth. Let the same pendulum make n' oscillations in the time T, when at a height h above the surface of the earth; let r be the radius of the earth.

The force of gravity varies inversely as the square of the distance

from the centre of the earth. Therefore, if G be the force of gravity at the height h,

$$G : g :: \frac{1}{(r+h)^2} : \frac{1}{r^2},$$

or,
$$G = \frac{gr^2}{(r+h)^2}.$$

But $n : n' :: \sqrt{g} : \sqrt{G}$; therefore,

$$\frac{n}{n'} = \frac{r+h}{r};$$

$$\therefore \qquad h = \frac{(n-n')r}{n'},$$

$$= \frac{(n-n')r}{n} \text{ nearly.}$$

383. *To find the depth of a mine by observing the number of oscillations lost by a pendulum in a given period, upon the supposition that the earth is a sphere of uniform density.*

Let G be the force of gravity at the bottom of the mine, and h the depth of the mine. Since the earth is regarded as a sphere of uniform density, the force of gravity in its interior varies directly as the distance from the centre. Therefore,

$$\frac{G}{g} = \frac{r-h}{r},$$

Whence,
$$\frac{n'}{n} = \sqrt{\left(\frac{G}{g}\right)} = \sqrt{\frac{r-h}{r}};$$

$$\therefore \qquad \left(\frac{n'}{n}\right)^2 = \frac{r-h}{r},$$

or,
$$\frac{h}{r} = 1 - \left(\frac{n'}{n}\right)^2 = \frac{2(n-n')}{n} \text{ nearly.}$$

384. *To find the reaction of the curve when a particle moves over a fixed curve under the action of any force.*

Let N be the resolved part of the given force in the direction of the normal, and R the reaction at any point. Then the three forces N, R, and the centrifugal force must be in equilibrium; and, since they act along the same line, their algebraic sum must be equal to zero. Let the direction of the centrifugal force be taken as the positive direction, then R is positive when the particle moves over the convex side of the curve, and negative when the particle moves over the concave side; hence

$$\pm \mathrm{R} + \mathrm{N} + \frac{v^2}{\rho} = 0.$$

In applying this formula, the student must remember that N is negative when non-concurrent with the centrifugal force.

In a similar way the tension in rods or cords may be found, when a body moving in a curve is connected with a fixed point by a rod or cord.

The following are examples in illustration:—

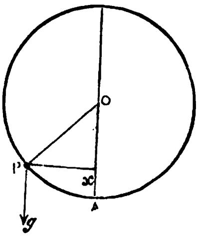
Ex. 1. A body, whose mass is m, is connected with a fixed point by an inextensible cord or rod, supposed to be without weight, and moves with a uniform velocity v about the point as a centre; to find the tension in the cord or rod.

As explained in the note to Art. 357, the pressure, when a force f acts upon a mass m, may be represented by the product mf. In the present case, f is equal to the centrifugal force; that is, to $\frac{v^2}{r}$, r being the length of the rod or cord;

$$\therefore \qquad \text{tension required} = \frac{mv^2}{r}.$$

Ex. 2. A body connected with a fixed centre by an inextensible rod, supposed to be without weight, moves over the arc of a circle under the action of gravity; to find the tension in the rod at any point.

Let m be the mass of the body, r the length of the rod, h the height above the lowest point A of the point from which the body fell, and x the height above A when in any position P.



If v be the velocity of the body at P, then

$$v^2 = 2g(h-x); \text{ and } N = \frac{g(r-x)}{r},$$

$$\therefore \qquad R = \frac{g(r-x)}{r} + \frac{2g(h-x)}{r},$$

$$= \frac{g(r+2h-3x)}{r}.$$

$$\text{Tension required} = \frac{mg(r+2h-3x)}{r}.$$

The tension in the rod will be zero when $x = \frac{1}{3}(r+2h)$; and if the body fall from the highest point of the circle, the tension is zero when $x = \frac{5r}{3}$.

Ex. 3. A particle moves from rest down the convex side of a circle whose plane is vertical, from a given point in its circumference, to find where it will leave the curve.

Let A be the highest point of the circle, B the point from which the particle starts, and P the position of the particle at any instant. Let a and θ be the angles made with the normal through A by the normals through B and P. Then if R be the reaction at P, and r the radius of the circle,

$$R = g\cos\theta - \frac{v^2}{r};$$

but, Art. 376, $\qquad v^2 = 2g(r\cos a - r\cos\theta).$

$$\therefore \qquad R = g(3\cos\theta - 2\cos a).$$

Hence, R is equal to zero, or the particle will leave the curve when

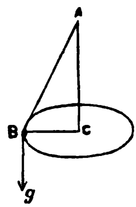
$$\cos\theta = \tfrac{2}{3}\cos a.$$

Or, if h be the height of B from the horizontal diameter, and x the height of P when the particle leaves the curve,

$$\begin{aligned} x &= r \cos \theta, \\ &= \tfrac{2}{3} r \cos \alpha, \\ &= \tfrac{2}{3} h. \end{aligned}$$

Ex. 4. A body fastened by an inextensible cord to a fixed point moves uniformly in a horizontal circle; to find the velocity at any point.

Let m be the mass of the body, and c the length of the cord. Let B be any position of the body, C the centre of the horizontal circle, and the angle BAC $= \alpha$. In any position the body is in equilibrium under three forces; the centrifugal force, the force of gravity, and the tension in the cord. Therefore, by triangle of forces,



$$mg : \frac{mv^2}{\mathrm{BC}} :: \sin \mathrm{ABC} : \sin \mathrm{BAC}.$$

But BC $= c \sin \alpha$, $\sin \mathrm{ABC} = \cos \alpha$, and $\sin \mathrm{BAC} = \sin \alpha$;

$$\therefore \qquad v^2 = \frac{gc \sin^2 \alpha}{\cos \alpha}.$$

Cor. If T be the time of describing an entire revolution, since space = velocity × time,

$$\begin{aligned} \mathrm{T} &= \frac{2\pi c \sin \alpha}{v}, \\ &= 2\pi \sqrt{\left(\frac{c \cos \alpha}{g}\right)}, \\ &= 2\pi \sqrt{\left(\frac{\mathrm{AC}}{g}\right)}. \end{aligned}$$

Hence the time of a complete revolution is independent of the length of the cord, and depends only upon the height of the fixed point above the plane of the circle.

EXAMPLES.

1. A perforated ball moves over a smooth circular ring whose plane is vertical, the ball is projected upward from one extremity of the horizontal diameter, with such a velocity that it will come to rest at the highest point of the ring, show that the reaction is zero when the ball has moved over an arc subtending an angle whose sine is $\frac{2}{3}$.

2. In the preceding, with what velocity must the ball be projected that the reaction may be zero, when the ball reaches the summit?

The required velocity is equal to $\sqrt{(3gr)}$.

3. A circle is placed with its plane vertical, with what velocity must a particle be projected vertically upwards, from one extremity of the horizontal diameter along the interior of the circle, that after quitting the curve it may pass through the centre?

The required velocity is equal to $\sqrt{(gr\sqrt{3})}$.

4. A body moves from rest over a circular quadrant from its highest to its lowest point, the radius at the highest point being horizontal and at the lowest vertical, show that on reaching the lowest point the reaction is equal to three times the force of gravity.

5. In the preceding, if the radius at the highest point be vertical and at the lowest horizontal, the reaction at the lowest point is equal to twice the force of gravity.

6. A particle moves from rest over the convex side of an ellipse

whose major axis is vertical, from a given point in the curve, to find when it will leave the ellipse.

Let h be the height of the given point above the minor axis, x that of the point at which the particle will leave the ellipse, then the value of x is determined by the equation,

$$e^2x^3 - 3a^2x + 2a^2h = 0.$$

7. An ellipse is placed with its minor axis vertical, with what velocity must a particle be projected vertically from the extremity of the major axis along the interior of the ellipse, that after quitting the curve it may pass through the centre?

Let a and b be the semi-axes of the ellipse, the required velocity will be equal to

$$\left\{\frac{g(a^2+8b^2)}{3b\sqrt{3}}\right\}^{\frac{1}{2}}.$$

The centre of pressure of a plane area is the pt in the ~~area~~ area at which the resultant pressure may be supposed to act

HYDROSTATICS.

CHAPTER IX.

ON THE CENTRE OF PRESSURE.

385. The pressure of a fluid upon any surface varies, as already seen, with the depth of each point below the surface of the fluid. The point in the surface through which the resultant of these several pressures passes is called the *centre of pressure.* When the surface in contact with the fluid is a plane surface, the pressures at the different points are all perpendicular to the surface, and hence are parallel forces. Hence, if the pressures on the several parts of a surface be P_1, P_2, &c. and k_1, k_2, &c. be the distances of the centres of pressure of these parts from the surface of the fluid, and k the distance of the centre of pressure of the whole plane from the surface of the fluid (Art. 52),

$$k = \frac{P_1 k_1 + P_2 k_2 + \&c.}{P_1 + P_2 + \&c.}.$$

But if m_1, m_2, &c. be the areas of the several parts of the plane, and h_1, h_2, &c. the distances of their *centres of gravity* below

whose major axis is vertical, from a given point in the curve, to find when it will leave the ellipse.

Let h be the height of the given point above the minor axis, x that of the point at which the particle will leave the ellipse, then the value of x is determined by the equation,

$$e^2x^3 - 3a^2x + 2a^2h = 0.$$

7. An ellipse is placed with its minor axis vertical, with what velocity must a particle be projected vertically from the extremity of the major axis along the interior of the ellipse, that after quitting the curve it may pass through the centre?

Let a and b be the semi-axes of the ellipse, the required velocity will be equal to

$$\left\{\frac{g(a^2+8b^2)}{3b\sqrt{3}}\right\}^{\frac{1}{2}}.$$

The centre of pressure of a plane area [illegible] [illegible] pressure may be [illegible]

HYDROSTATICS.

CHAPTER IX.

ON THE CENTRE OF PRESSURE.

385. The pressure of a fluid upon any surface varies, as already seen, with the depth of each point below the surface of the fluid. The point in the surface through which the resultant of these several pressures passes is called the *centre of pressure.* When the surface in contact with the fluid is a plane surface, the pressures at the different points are all perpendicular to the surface, and hence are parallel forces. Hence, if the pressures on the several parts of a surface be P_1, P_2, &c. and k_1, k_2, &c. be the distances of the centres of pressure of these parts from the surface of the fluid, and k the distance of the centre of pressure of the whole plane from the surface of the fluid (Art. 52),

$$k = \frac{P_1k_1 + P_2k_2 + \&c.}{P_1 + P_2 + \&c.}.$$

But if m_1, m_2, &c. be the areas of the several parts of the plane, and h_1, h_2, &c. the distances of their *centres of gravity* below

the surface of the fluid, and w the weight of a unit of the fluid, $P_1 = wm_1h_1$, $P_2 = wm_2h_2$, &c.;

$$\therefore \qquad k = \frac{m_1h_1k_1 + m_2h_2k_2 + \&c.}{m_1h_1 + m_2h_2 + \&c.}.$$

If the parts into which the plane is divided be indefinitely small rectangles, having one pair of sides parallel to the surface of the fluid, the centres of pressure and the centres of gravity may be regarded as coincident. Whence,

$$k = \frac{m_1h_1^2 + m_2h_2^2 + \&c.}{m_1h_1 + m_2h_2 + \&c.},$$

or, as it may be conveniently written,

$$k = \frac{\Sigma . mh^2}{\Sigma . mh}.$$

Cor. If the areas of the several parts are equal, the last expression becomes

$$k = \frac{h_1^2 + h_2^2 + \&c.}{h_1 + h_2 + \&c.}$$

$$= \frac{\Sigma . h^2}{\Sigma . h}.$$

386. *To find the centre of pressure of any parallelogram, one of whose sides is coincident with the surface of the fluid.*

Let h be the depth of the lower side of the parallelogram. Let the parallelogram be divided by lines parallel to the surface into n equal parallelograms, n being an indefinitely large number. Since these parallelograms are indefinitely small, the depth of their

centres of gravity may be taken as equal to that of their lower sides. Hence,

$$k = \frac{\left(\frac{h}{n}\right)^2 + \left(\frac{2h}{n}\right)^2 + \left(\frac{3h}{n}\right)^2 + \dots\dots + \left(\frac{nh}{n}\right)^2}{\frac{h}{n} + \frac{2h}{n} + \frac{3h}{n} + \dots\dots + \frac{nh}{n}},$$

$$= h\frac{\dfrac{1 + 2^2 + \dots\dots + n^2}{n^3}}{\dfrac{1 + 2 + \dots\dots + n}{n^2}},$$

$$= \frac{2h}{3}, \text{ since } n \text{ is large (Art. 317),}$$

or the centre of pressure is at two-thirds of the depth.

Hence, the staves by a cylindrical barrel, containing any liquid, may be kept in their place by a single hoop placed at two-thirds of the depth of the fluid.

387. *To determine the centre of pressure of a parallelogram immersed to any depth, but having one of its sides parallel to the fluid.*

Let ABCD be any parallelogram, having its lower side AB parallel to the surface of the fluid. Let h be the depth of AB below the surface, and h' the depth of the opposite side DC. Produce the sides AD, BC to meet the surface of the fluid in K and L. Let $AB = a = CD$, and let θ be the inclination of the plane to the vertical line.

$$\text{Area of ABLK} = \frac{ah}{\cos\theta};$$

$$\therefore \qquad \text{pressure on ABLK} = \frac{ah}{\cos\theta}\cdot\frac{h}{2}.$$

Similarly, $\qquad \text{pressure on DCLK} = \dfrac{ah'}{\cos\theta}\cdot\dfrac{h'}{2}.$

Hence, if k be the depth of the centre of pressure on ABCD,

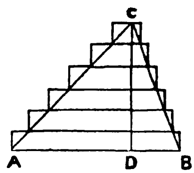
$$k = \frac{\dfrac{ah}{\cos\theta}\cdot\dfrac{h}{2}\cdot\dfrac{2h}{3} - \dfrac{ah'}{\cos\theta}\cdot\dfrac{h'}{2}\cdot\dfrac{2h'}{3}}{\dfrac{ah}{\cos\theta}\cdot\dfrac{h}{2} - \dfrac{ah'}{\cos\theta}\cdot\dfrac{h'}{2}},$$

$$= \frac{2}{3}\cdot\frac{h^3 - h'^3}{h^2 - h'^2},$$

$$= \frac{2}{3}\cdot\frac{h^2 + hh' + h'^2}{h + h'}.$$

388. *To find the centre of pressure of a triangular plane, whose vertex is on the surface of the fluid, and whose base is parallel to the surface of the fluid.*

Let ABC be the triangle, and AB its base. Draw CD perpendicular to the base, and divide it into n equal parts. Through these points of division draw lines parallel to AB, and upon these lines as bases describe rectangles, as in the figure. As n increases, the sum of the pressures on the rectangles approaches to the pressure on the triangle.



Let AB $= a$, and let h be the depth of AB below the surface of the fluid. Then the base of the first rectangle is equal to $\frac{a}{n}$ the base of the second rectangle to $\frac{2a}{n}$, and so on;

$$\therefore \quad \text{area of first rectangle} = \frac{a}{n}\cdot\frac{\text{CD}}{n},$$

$$\text{second} \quad ,, \quad = \frac{2a}{n}\cdot\frac{\text{CD}}{n},$$

$$\text{third} \quad ,, \quad = \frac{3a}{n}\cdot\frac{\text{CD}}{n},$$

&c. &c.

The depth of the centre of gravity of the first rectangle may be taken as equal to $\frac{h}{n}$, of the second $\frac{2h}{n}$, of the third $\frac{3h}{n}$, and so on. Let k be the depth of the centre of pressure of the triangle, then

$$k = \frac{\frac{a\text{CD}}{n^2}\cdot\left(\frac{h}{n}\right)^2 + \frac{2a\text{CD}}{n^2}\cdot\left(\frac{2h}{n}\right)^2 + \&\text{c.} + \frac{na\text{CD}}{n^2}\left(\frac{nh}{n}\right)^2}{\frac{a\text{CD}}{n^2}\cdot\frac{h}{n} + \frac{2a\text{CD}}{n^2}\cdot\frac{2h}{n} + \&\text{c.}\, . + \frac{n\,.\,a\text{CD}}{n^2}\cdot\frac{nh}{n}},$$

$$= h\frac{\frac{1^3+2^3+\ldots\ldots+n^3}{n^4}}{\frac{1^2+2^2+\ldots\ldots+n^2}{n^3}},$$

$$= \frac{3h}{4}. \quad \text{(Art. 317.)}$$

389. *To find the centre of pressure of a triangular plane whose base is on the surface of the fluid.*

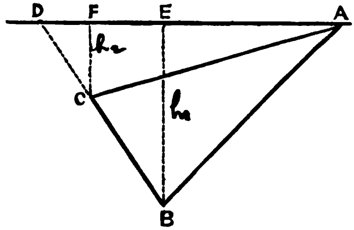
Let h be the depth of the vertex, k the depth of the centre of pressure. The given triangle is the half of a parallelogram, one of whose sides is coincident with the surface of the fluid, and whose opposite side is at depth h. The centre of pressure of this parallelogram is known from Art. 386. The other half of the parallelogram is a triangle whose vertex is on the surface, and whose base is parallel to the surface; its centre of pressure is therefore known by the preceding Article. Then, if A be the area of the given triangle,

$$k = \frac{\frac{2\text{A}h}{2}\cdot\frac{2h}{3} - \frac{2\text{A}h}{3}\cdot\frac{3h}{4}}{\frac{2\text{A}h}{2} - \frac{2\text{A}h}{3}},$$

$$= h\,.\,\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{2}-\frac{1}{3}} = \tfrac{1}{2}h.$$

390. *To find the centre of pressure of any triangular plane whose vertex is on the surface of the fluid.*

Let ABC be the given plane, and let h_1, h_2, be the depths of B and C respectively.



Let BC produced meet the surface of the fluid in D. Draw BE, CF, perpendicular to AD. Then, if α be the inclination of the plane of ABC to the surface of the fluid,

$$\text{BE} \sin \alpha = h_1,$$

and $$\text{CF} \sin \alpha = h_2.$$

The given plane is equal to the difference of the two triangles ABD and ACD.

The area of ABD is equal to $\frac{1}{2}$ AD × BE, or $\dfrac{\text{AD} \, . \, h_1}{2 \sin \alpha}$; the depth of its centre of gravity is $\frac{1}{3}h_1$; and the depth of its centre of pressure is (Art. 389) $\frac{1}{2}h_1$.

The area of ACD is equal to $\frac{1}{2}$ AD × CF, or $\dfrac{\text{AD} \, . \, h_2}{2 \sin \alpha}$; the depth of its centre of gravity is $\frac{1}{3}h_2$; and the depth of its centre of pressure is $\frac{1}{2}h_2$.

Therefore, if k be the depth of the centre of pressure of ABC,

$$k = \frac{\dfrac{\text{AD} \, . \, h_1}{2 \sin \alpha} \cdot \dfrac{h_1}{3} \cdot \dfrac{h_1}{2} - \dfrac{\text{AD} \, . \, h_2}{2 \sin \alpha} \cdot \dfrac{h_2}{3} \cdot \dfrac{h_2}{2}}{\dfrac{\text{AD} \, . \, h_1}{2 \sin \alpha} \cdot \dfrac{h_1}{3} - \dfrac{\text{AD} \, . \, h_2}{2 \sin \alpha} \cdot \dfrac{h_2}{3}},$$

$$= \frac{1}{2} \cdot \frac{h_1^3 - h_2^3}{h_1^2 - h_2^2},$$

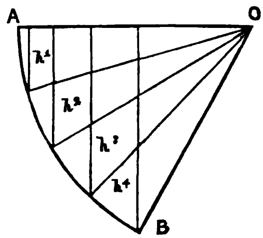
$$= \frac{1}{2} \cdot \frac{h_1^2 + h_1 h_2 + h_2^2}{h_1 + h_2}.$$

COR. If h_2 is nearly equal to h_1, the depth of the centre of pressure is nearly

$$\frac{3h_1}{4}.$$

391. *To find the depth of the centre of pressure of a circular sector immersed vertically with one of its radii upon the surface of the fluid.*

Let AOB be the sector, having the angle AOB equal to α. Divide the sector into n equal sectors, n being indefinitely large. These sectors may be regarded as small triangles.



Let h_1 be the depth of the lowest vertex of the first triangle, h_2, that of the second, and so on. Then the depths of the centre of gravity of the different triangles are respectively

$$\frac{2h_1}{3}, \quad \frac{2h_2}{3}, \quad \ldots\ldots \frac{2h_n}{3},$$

and the depths of the centres of pressure (Art. 390, Cor.)

$$\frac{3h_1}{4}, \quad \frac{3h_2}{4}, \quad \ldots\ldots \frac{3h_n}{4}.$$

Therefore, if k be the depth of the centre of pressure of the sector,

$$k = \frac{\frac{1}{2}(h_1^2 + h_2^2 + \ldots\ldots + h_n^2)}{\frac{2}{3}(h_1 + h_2 + \ldots\ldots + h_n)}.$$

But if a be the radius of the sector,

$$h_1 = a \sin\frac{\alpha}{n},$$

$$h_2 = a \sin\frac{2\alpha}{n},$$

$$h_n = a \sin\frac{n\alpha}{n}.$$

Therefore, substituting these values,

$$k = \frac{3a}{4} \cdot \frac{\sin^2\frac{a}{n} + \sin^2\frac{2a}{n} + \ldots\ldots + \sin^2\frac{na}{n}}{\sin\frac{a}{n} + \sin\frac{2a}{n} + \ldots\ldots + \sin\frac{na}{n}},$$

$$= \frac{3a}{8} \cdot \frac{n - \left(\cos\frac{2a}{n} + \cos\frac{4a}{n} + \ldots + \cos\frac{2na}{n}\right)}{\sin\frac{a}{n} + \sin\frac{2a}{n} + \ldots\ldots + \sin\frac{na}{n}},$$

$$= \frac{3a}{8} \cdot \frac{n - \dfrac{\cos\left(\dfrac{n+1}{n} \cdot a\right)\sin a}{\sin\dfrac{a}{n}}}{\dfrac{\sin\left(\dfrac{n+1}{n} \cdot \dfrac{a}{2}\right)\sin\dfrac{a}{2}}{\sin\dfrac{a}{2n}}},$$

$$= \frac{3a}{8} \cdot \frac{a - \cos a \sin a}{1 - \cos a}, \text{ since } n \text{ is large.}$$

392. Cor. I. If the sector be a quadrant, $a = \frac{\pi}{2}$, and therefore

$$k = \frac{3\pi a}{16}.$$

393. Cor. II. If the sector be a semi-circle, $a = \pi$, and therefore

$$k = \frac{3\pi a}{16}.$$

394. Cor. III. Hence, also, if a sector be immersed with its

plane vertical, its centre upon the surface, and its axis inclined to the surface at an angle θ; then a being the angle of the sector,

$$k = \frac{3a}{16} \cdot \frac{2a - \sin(2\theta + a) + \sin(2\theta - a)}{\cos(\theta - \frac{a}{2}) - \cos(\theta + \frac{a}{2})}.$$

395. Cor. iv. If a semi-circle, having its diameter on the surface of a fluid, be divided into small rectangles by lines parallel to the surface, and if m denote the area of one of these, and h the depth of its centre of gravity, then (Art. 385),

$$k = \frac{\Sigma \,.\, mh^2}{\Sigma \,.\, mh}.$$

But, $k = \frac{3\pi a}{16}$, and $\Sigma \,.\, mh$, or the sum of the moments of the several rectangles about the diameter is equal to the moment of the whole semi-circle; that is, to $\frac{\pi a}{2} \times \frac{4a}{3\pi}$ or $\frac{2}{3}a^3$. Substituting these values, we have

$$\Sigma \,.\, mh^2 = \frac{\pi a^4}{8}.$$

But the centres of gravity of the rectangles all lie in the vertical radius, and consequently their distances from the surface of the fluid are the same as their distances from the centre of the circle. Hence, if a semi-circle be divided into small rectangles by lines parallel to its diameter, the sum of the product of each rectangle into the square of the distance of its centre of gravity from the centre of the circle is equal to

$$\frac{\pi a^4}{8}.$$

Hence also the sum of a similar product for the whole circle is

$$\frac{\pi a^4}{4}.$$

396. *To find the centre of pressure of a circle just immersed in a fluid with its plane vertical.*

Let the circle be divided into small rectangles by lines parallel to the surface; then, if m_1, m_2, &c. denote their areas, h_1, h_2, &c. the depths of their centres of gravity, and k the depth of the centre of pressure,

$$k = \frac{\Sigma . mh^2}{\Sigma . mh}.$$

But, since the centres of gravity of the rectangles lie in the vertical diameter, their distances from the surface of the fluid are the same as their distances from the highest point of the circle; and, consequently, if r_1, r_2, &c. denote their distances from the centre of the circle (Art. 326),

$$\Sigma . mh^2 = \Sigma . mr^2 + a^2 . \Sigma m.$$

But, by preceding Article, $\Sigma . mr^2 = \frac{\pi a^4}{4}$, and Σm, or the area of the circle, is equal to πa^2; therefore

$$\Sigma . mh^2 = \frac{5\pi a^4}{4}.$$

And $\Sigma . mh$ is equal to the moment of the circle about the surface, or to $\pi a^2 \times a$; therefore

$$k = \frac{\Sigma . mh^2}{\Sigma . mh},$$

$$= \frac{5\pi a^4}{4\pi a^3};$$

$$= \frac{5a}{4};$$

or the distance of the centre of pressure below the centre of the circle is equal to one-fourth of the radius.

397. *To determine the variation in the position of the centre of pressure of any plane figure corresponding to a given variation in the*

position of the centre of gravity, the same lines in the figure being parallel to the surface of the fluid in each case.

Let the given figure be divided into small rectangles by lines parallel to the surface of the fluid. Let m be the magnitude of any one of these rectangles, and h the distance of its centre of gravity from the line in which the plane of the given figure intersects the surface of the fluid. Let h_1, k_1 be the distances of the centres of gravity and pressure of the given figure from the same line, and let α be the inclination of the plane to the surface of the fluid; then $h_1 \sin \alpha$, $k_1 \sin \alpha$ are the depths of the centres of gravity and pressure; and therefore

$$k_1 \sin \alpha = \frac{\Sigma . mh^2 \sin^2 \alpha}{\Sigma . mh \sin \alpha},$$

$$= \frac{\sin^2 \alpha \, \Sigma . mh^2}{\sin \alpha \, \Sigma . mh};$$

$$\therefore \qquad k_1 = \frac{\Sigma . mh^2}{\Sigma . mh}.$$

Also,
$$h_1 = \frac{\Sigma . mh}{\Sigma . m};$$

$$\therefore \qquad h_1 k_1 = \frac{\Sigma . mh^2}{\Sigma . m}.$$

But if r be the distance of the centre of gravity of any small rectangle from a line through the centre of gravity of the figure, parallel to the surface of the fluid, then (Art. 326, Note,)

$$\Sigma . mh^2 = \Sigma . mr^2 + h_1^2 \Sigma m.$$

Consequently,

$$h_1 k_1 = \frac{\Sigma . mr^2}{\Sigma m} + h_1^2,$$

and therefore, removing h_1^2 to the other side of the equation,

$$h_1 (k_1 - h_1) = \frac{\Sigma . mr^2}{\Sigma m}. \qquad \text{(i.)}$$

In like manner, if h_2, k_2 be the distances of the centre of gravity and the centre of pressure from the surface line in the second position,

$$h_2(k_2 - h_2) = \frac{\Sigma . mr^2}{\Sigma m}. \qquad \text{(ii.)}$$

But, by hypothesis, the line through the centre of gravity, parallel to the surface of the fluid, is the same in both cases, and consequently $\Sigma . mr^2$ is the same in equations i. and ii.; therefore,

$$h_2(k_2 - h_2) = h_1(k_1 - h_1).$$

Cor. If the figure is symmetrical about an axis, and the axis is at right angles to the surface line, then since both the centre of gravity and the centre of pressure are in the axis, the distance $k - h$ is in each case the distance of pressure from the centre of gravity, and hence the preceding theorem shows, that *if any symmetrical figure be immersed with its axis at right angles to the surface line, the product of the distance of the centre of gravity from the surface line into the distance of the centre of pressure from the centre of gravity is invariable.*

EXAMPLES.

1. Find the centre of pressure of a triangular plane immersed to any depth, having its base parallel to the surface of the fluid, and lower than the vertex.

Let c be the depth of the vertex, and $c + h$ that of the base; then the depth of the centre of pressure is equal to

$$\frac{6c^2 + 8ch + 3h^2}{6c + 4h}.$$

2. Find the centre of pressure of a square plane, having one of its corners upon the surface of the fluid, and one of its diagonals vertical.

Let a be the side of the square, then the depth of the centre of pressure is equal to

$$\frac{7a\sqrt{2}}{12}.$$

3. Find the centre of pressure of a hexagonal plane immersed vertically, with one of its sides upon the surface of the fluid.

Let a be a side of the hexagon, then the depth of the centre of pressure is equal to

$$\frac{23a\sqrt{3}}{36}.$$

4. Find the centre of pressure of a trapezium immersed with its base upon the surface of the fluid, and its parallel sides perpendicular to the surface.

Let h_1 and h_2 be the lengths of the parallel sides, then the depth of the centre of pressure is equal to

$$\frac{1}{2}\frac{(h_1+h_2)\,(h_1^2+h_2^2)}{h_1^2+h_1h_2+h_2^2}.$$

5. A conical vessel, filled with a liquid, is placed with its vertex upwards, and its axis inclined to the vertical line at an angle equal to half the angle of the cone; determine the distance of the centre of pressure of the base from the centre of the base.

Let $2a$ be the angle of the cone, and r the radius of the base; then the distance required is equal to

$$\frac{r\tan^2 a}{4}.$$

6. An octagonal plate is immersed in a fluid, so that one of its sides is upon the surface; find the distance of the centre of pressure from the centre of the plate.

Let a be the side of the plate, then the distance required is equal to

$$\frac{a(1+2\sqrt{2})}{12}.$$

CHAPTER X.

ON THE EQUILIBRIUM OF FLOATING BODIES.

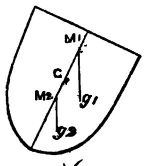
398. It has already been seen that, when a floating body is in equilibrium, the weight of the body acting vertically downwards at its centre of gravity is equal and opposite to the weight of the fluid displaced, acting at the point which was the centre of gravity of this portion of the fluid before displacement. If then a vertical line passing through this last-mentioned point be called the *line of thrust*, one of the conditions of the equilibrium of a floating body will be, that the line of thrust pass through its centre of gravity.

399. A body at rest is said to be in *stable* equilibrium, if after receiving a slight displacement it tends to return to its former position, but *unstable* if it tends to recede from it. If neither of these happen—that is, if equilibrium be not destroyed by the displacement, the body is said to be in neutral equilibrium.

400. If a body in equilibrium be displaced through an indefinitely small angle, the point in which the lines of thrust before and after displacement intersect one another is termed the *metacentre.*

401. *The equilibrium of a floating body is stable, unstable, or neutral, according as the metacentre is above, below, or coincident with the centre of gravity.*

Let G be the centre of gravity of a floating body, and M_1GM_2 the line of thrust when at rest. Let g_1M_1 be the line of thrust after displacement, then M_1 is the metacentre, and falls above G. The body is then acted upon by two equal forces, the weight of the body acting vertically downwards at G, and the resultant pressure of the fluid acting upwards at g_1.



These forces will rise to a motion of rotation, such as tends to restore the line M_1GM_2 to the vertical position, and consequently the equilibrium is stable.

Again, let g_2M_2 be the line of thrust after displacement, so that the metacentre is below G, then the two forces acting upon the body tend to produce rotation in a direction contrary to that in the former case, or to cause M_1GM_2 to recede from the vertical position; the equilibrium is therefore unstable.

If the line of thrust after displacement pass through G, the body will still remain at rest, and therefore the equilibrium is neutral.

402. *To find the depth of the immersed part of any floating body when its form is similar to that of the whole body.*

Let V be the volume of the given body, V′ that of the part immersed, and s the ratio of the specific gravity of the body to that of the fluid. Let w be the weight of a unit of the fluid, then the weight of the body is equal to

$$Vsw,$$

and the weight of the fluid displaced is equal to

$$V'w;$$

and therefore, by Art. 216,

$$V'w = Vsw,$$

and hence,

$$\frac{V'}{V} = s.$$

Let h be the vertical height of the floating body, and h' that of the part immersed. Then, if the body be a plate of uniform thickness,

$$\frac{h'^2}{h^2} = \frac{V'}{V} = s,$$

and therefore, in this case,

$$h' = h\sqrt{s}.$$

Secondly; if the body be not of uniform thickness, then, since by hypothesis the floating body and the part immersed are similar solids,

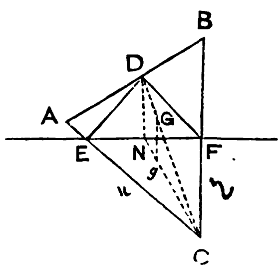
$$\frac{h'^3}{h^3} = \frac{V'}{V} = s,$$

and therefore, in this case,

$$h' = h\sqrt[3]{s}.$$

403. *To find the positions of equilibrium of a triangular plate floating vertically in any fluid.*

Let ABC be the triangular plate, G its centre of gravity, and s the ratio of the specific gravity of the floating body to that of the fluid.



First; let one vertex only be immersed, as in the figure, and let g be the centre of gravity of the displaced fluid. Then, by the conditions of equilibrium, gG must be vertical, and the weight of the fluid displaced must equal the weight of the floating body.

Let AC $= a$, BC $= b$, EC $= x$, and FC $= y$, and let C represent the angle ACB;

$$\text{area of ABC} = \tfrac{1}{2}ab \sin C,$$
$$\text{area of EFC} = \tfrac{1}{2}xy \sin C.$$

Hence, if w be the weight of a unit of the fluid, and s the ratio of the specific gravity of the body to that of the fluid,

$$\text{weight of ABC} = \tfrac{1}{2} swab \sin C,$$

$$\text{weight of displaced fluid} = \tfrac{1}{2} wxy \sin C.$$

Therefore, $$wxy = swab,$$

$$xy = sab. \qquad \text{(i.)}$$

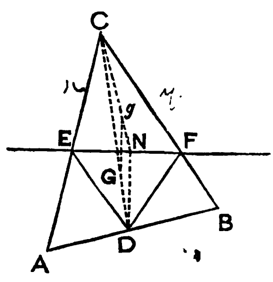
Since g is the centre of gravity of EFC, N is the bisection of EF, and $\mathrm{N}g = \frac{1}{3}\,\mathrm{NC}$. In like manner, D is the bisection of AB, and $\mathrm{DG} = \frac{1}{3}\,\mathrm{DC}$. Hence, in the triangle DCN, DG : DC : : Ng : NC, and therefore DN and Gg are parallel. Consequently, DN is vertical, or at right angles to EF, and therefore the lines DE and DF are equal. page 212. ln. 12.

But $$\mathrm{DE}^2 = \mathrm{DC}^2 + x^2 - 2x\mathrm{DC} \cos \mathrm{ACD},$$

and $$\mathrm{DF}^2 = \mathrm{DC}^2 + y^2 - 2y\mathrm{DC} \cos \mathrm{BCD};$$

$$\therefore\ x^2 - 2x\mathrm{DC} \cos \mathrm{ACD} = y^2 - 2y\mathrm{DC} \cos \mathrm{BCD}. \qquad \text{(ii.)}$$

The equations i. and ii. are sufficient for determining the values of x and y, or the positions of equilibrium, when C only is immersed. And, in like manner, the positions of equilibrium may be found when either of the vertices is immersed singly.



Secondly; let one vertex only, C, be without the fluid. Then, as before, Gg must be vertical; for since G is the centre of gravity of the whole figure ABC, and g of the part EFC, the centre of gravity of the remainder ABFE must lie in the line joining Gg. Consequently, if, as before, $\mathrm{CE} = x$, and

CF $= y$, equation ii. applies to this case also. To obtain the other equation,

$$\text{ABFE} = \text{ABC} - \text{EFC},$$

$$= \tfrac{1}{2}(ab - xy) \sin \text{C};$$

$\therefore$ weight of displaced fluid $= \frac{1}{2}w(ab - xy) \sin \text{C}$;

$$\therefore \quad w(ab - xy) \sin \text{C} = swab \sin \text{C},$$

$$ab - xy = sab,$$

$$xy = (1 - s)ab.$$

404. The equations deduced in the preceding Article admit of easy solution, if the vertex to which they apply be the vertex of an isosceles triangle. Let this angle be 2α, and let each of the equal sides be a; then DC $= a \cos \alpha$, and the equations i. and ii. become

$$xy = sa^2,$$

and $$x^2 - 2ax \cos^2 \alpha = y^2 - 2ay \cos^2 \alpha;$$

$$\therefore \quad x^2 - y^2 - 2a \cos^2 \alpha \, (x - y) = 0,$$

$$\therefore \quad (x - y)\,(x + y - 2a \cos^2 \alpha) = 0.$$

This equation is satisfied when either of its factors vanishes. Let the first factor vanish, then

$$x - y = 0;$$

$$\therefore \quad x = y = a\sqrt{s},$$

which gives the position of the triangle when floating with DC vertical. Let the second factor vanish, then

$$x + y - 2a \cos^2 \alpha = 0.$$

Substituting the value of y, from the equation $xy = sa^2$,

$$x^2 - 2ax \cos^2 \alpha + sa^2 = 0;$$

$$\therefore \quad x = a \cos^2 \alpha \pm a\sqrt{(\cos^4 \alpha - s)}.$$

If s be greater than $\cos^4 a$, this expression involves an impossible quantity, and the only position of equilibrium will be that already determined. If $s = \cos^4 a$, the expression under the radical vanishes, and we have

$$x = a \cos^2 a = a\sqrt{s};$$

$$\therefore \qquad y = \frac{s \,.\, a^2}{a \,.\, \cos^2 a} = a\sqrt{s},$$

which gives the same position as before. If s be less than $\cos^4 a$, the equations will give two real values for x and y, and there may consequently be two positions of equilibrium besides that already determined.

If the vertex be without the fluid, the equations for the determination of the position of equilibrium become

$$xy = (1 - s)\, a^2,$$

and $$(x - y)(x + y - 2a \cos^2 a) = 0.$$

Whence $$x = a\sqrt{(1 - s)},$$

and $$x = a \cos^2 a \pm a\sqrt{\{\cos^4 a - (1 - s)\}},$$

which may give three positions of equilibrium, if $1 - s$ be less than $\cos^4 a$; that is, if s be greater than $1 - \cos^4 a$.

405. *To find the greatest number of positions in which an equilateral triangular plate will float in a given fluid.*

Since the plate is equilateral, $a = 30°$, and therefore $\cos^2 a = \frac{3}{4}$. Let $x_1, x_2, x_3, y_1, y_2, y_3$ be the different possible values of x and y when any one vertex is immersed; then

$$x_1 = \qquad a\sqrt{(s)} \qquad = y_1,$$

$$x_2 = a\,\{\tfrac{3}{4} + \sqrt{(\tfrac{9}{16} - s)}\} = y_3,$$

$$x_3 = a\,\{\tfrac{3}{4} - \sqrt{(\tfrac{9}{16} - s)}\} = y_2.$$

In order that x_2, x_3, y_2, y_3 may be possible, s must be less than $\frac{9}{16}$. The nature of the problem also requires that neither x nor y shall be greater than a. No value below $\frac{9}{16}$ given to s will make either the first or third expression greater than a. That the second expression be not greater than a, s must be so taken that

$$\tfrac{3}{4} + \sqrt{(\tfrac{9}{16} - s)} \text{ is not greater than } 1,$$

$$\text{or,} \qquad \tfrac{9}{16} - s \quad \text{,,} \quad \text{,,} \quad \tfrac{1}{16},$$

$$\text{or,} \qquad s \text{ is not less than } \tfrac{1}{2}.$$

If s be less than $\frac{1}{2}$, x_2 and y_3 are both greater than a, and therefore the positions denoted by x_2, y_2, and by x_3, y_3 are inadmissible.

Again, let x_4, y_4, x_5, y_5, x_6, y_6, be the possible values of x and y, when any vertex is singly above the fluid. Then

$$x_4 = \quad a\sqrt{(1 - s)} \quad = y_4,$$

$$x_5 = a\left\{\tfrac{3}{4} + \sqrt{(s - \tfrac{7}{16})}\right\} = y_6,$$

$$x_6 = a\left\{\tfrac{3}{4} - \sqrt{(s - \tfrac{7}{16})}\right\} = y_5.$$

In order that the second and third of these expressions may be possible, s must be greater than $\frac{7}{16}$, and that the second may not be greater than a, s must not be greater than $\frac{1}{2}$. If s be greater than $\frac{1}{2}$, x_5 and y_6 are both greater than a, and therefore the positions denoted by x_5, y_5, and by x_6, y_6, are inadmissible.

Hence we obtain the following results:—

First: If s be greater than unity, the body will not float at all.

Secondly: If s lie between 1 and $\frac{9}{16}$ the body will float in six positions; namely, when either vertex is singly above or below the surface, and the opposite side horizontal.

Thirdly: If s lie between $\frac{9}{16}$ and $\frac{1}{2}$, each vertex, when singly immersed, will rest in three positions, and when singly above the surface in one; namely, with the opposite side horizontal; there will therefore be twelve positions.

Fourthly: If s lie between $\frac{1}{2}$ and $\frac{7}{16}$, each vertex, when singly above the surface, will rest in three positions, and when singly

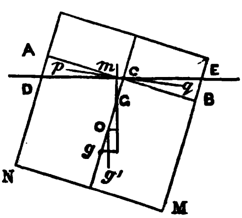
below in one; namely, with the base horizontal; there are therefore twelve positions of equilibrium.

Fifthly: If $s = \frac{1}{2}$, there are also twelve positions only. For though it might seem as if this would give three values for x, both when each vertex is singly below the surface and when above, it will be seen that when s has this value, either x or y, except when one side is horizontal, is equal to a, and consequently one vertex is on the surface. Six of the positions, therefore, having one vertex singly below will be identical with six having one vertex singly above. There will not then be eighteen positions of equilibrium, but twelve only.

Sixthly: If s be less than $\frac{7}{16}$, the body will float in six positions; namely, with either vertex singly above or below the surface, and the base horizontal.

406. *A square plate of uniform thickness and density floats vertically in a fluid with one side horizontal; to determine the position of the metacentre, and the nature of the equilibrium.*

Let $2a$ be the length of a side of the plate, and s the specific gravity of the plate relatively to that of the fluid. Let ABMN be the immersed part when the plate is in the position of equilibrium, then



$$ABMN = 4a^2s.$$

Let DEMN be the part immersed when the plate has been moved through an indefinitely small angle θ, in such a way that the quantity of fluid displaced after the disturbance is the same as that displaced before; and therefore

$$ACD = CEB.$$

Let G be the centre of gravity of the plate, g that of ABMN, and g' that of DEMN. Let the vertical line through g' meet the line gG in O, and then O is the metacentre.

If z be the distance of g' from the vertical line through G,

$$GO \sin \theta = z,$$

or, $$GO = \frac{z}{\sin \theta} = \frac{z}{\theta}, \text{ since } \theta \text{ is indefinitely small.}$$

The value of z may be found as follows:—

$$DEMN = ABMN - ACD + CEB.$$

Therefore the moment of DEMN about the vertical line through G is equal to the algebraic sum of the moments of these several parts.

The area of DEMN is equal to $4a^2s$, and the distance of its centre of gravity from the vertical line through G is by hypothesis z.

The area of ABMN is equal to $4a^2s$, and the distance of its centre of gravity from the vertical line through G is equal to $gG \cdot \theta$, or $a\theta\,(1 - s)$.

The area of the indefinitely small triangle ACD may be regarded as equal to that of a sector of the circle, whose centre is at C and radius CA or a; whence the area of ACD is equal to $\frac{1}{2}a^2\theta$, and the distance of its centre of gravity of ACD from C is equal to $\frac{2}{3}a$, and therefore its distance from the vertical line through G is $\frac{2}{3}a - m$C.

Similarly, the area of CEB is equal to $\frac{1}{2}a^2\theta$, and the distance of its centre of gravity from the vertical line through G is equal to $\frac{2}{3}a + m$C.

Therefore, assuming the left hand side of the vertical line through G to be the positive direction, and consequently giving the negative sign to the moment of CEB, we have

$$4a^2s \cdot z = 4a^3s \cdot \theta\,(1 - s) - \tfrac{1}{2}a^2\theta\,(\tfrac{2}{3}a - mC) - \tfrac{1}{2}a^2\theta\,(\tfrac{2}{3}a + mC),$$

and therefore,

$$z = a\theta \cdot \frac{6s\,(1 - s) - 1}{6s}.$$

Consequently,

$$GO = a \cdot \frac{6s\,(1 - s) - 1}{6s}.$$

If this value be positive, O falls below G, and the equilibrium is unstable. If it equal zero, O coincides with G, or the equilibrium is neutral. If it be negative, O falls above G, or the equilibrium is stable. Hence, the equilibrium is

unstable if $s^2 - s + \frac{1}{6}$ be negative,
neutral ,, ,, vanish,
stable ,, ,, positive.

The roots of this expression are

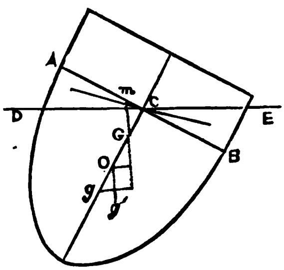
$$\frac{3+\sqrt{3}}{6} \text{ and } \frac{3-\sqrt{3}}{6}.$$

Hence the equilibrium is unstable if the value of s lie between these limits, neutral if it be equal to either of them, and stable if it lie without them.

407. The method pursued in the preceding Article may be easily adapted to the more general case of a plate of any symmetrical figure, and also to that of any solid of revolution.

Art. 407. Art. 409.

To find the metacentre of a plate of any symmetrical figure when floating with its axis vertical.



Let V be the magnitude of the part immersed. Let $2a$ be the line of floatation. Let A be the magnitude of the parts raised and depressed by the disturbance, which are evidently equal. Let p be the distance of the centre of gravity of either of these parts from C. Then, as in the preceding Article, taking the moments about the vertical line through G, we have

$$V\theta \,.\, GO = V\theta \,.\, gG - A\,(p - mC) - A\,(p + mC);$$

and therefore

$$GO = gG - \frac{2Ap}{V\theta}.$$

The parts raised and depressed may, as before, be regarded as small circular sectors, whose radius is a and angle θ. Hence, $A = \frac{1}{2}a^2\theta$, and $p = \frac{2}{3}a$, consequently,

$$GO = gG - \frac{2}{3} \cdot \frac{a^3}{V}.$$

408. Ex. *An isosceles triangular plate floats vertically, with its base horizontal and its vertex immersed; to determine the position of the metacentre, and the nature of the equilibrium.*

Let 2α be the vertical angle, h the vertical height of the triangle, and s the specific gravity of the plate relatively to the fluid.

In the position of equilibrium, the part immersed is also an isosceles triangle, whose vertical angle is 2α. Let h' be its height, then

$$V = h'^2 \tan \alpha,$$

$$a = h' \tan \alpha,$$

and $$gG = \tfrac{2}{3}(h - h').$$

Therefore, $$GO = \tfrac{2}{3}(h - h') - \tfrac{2}{3}h' \tan^2 \alpha,$$

$$= \tfrac{2}{3}\left(h - \frac{h'}{\cos^2 \alpha}\right).$$

But, $h' = h\sqrt{s}$ (Art. 402); therefore

$$GO = \frac{2h}{3}\left(1 - \frac{\sqrt{s}}{\cos^2 \alpha}\right).$$

Whence it follows that the equilibrium is unstable, neutral, or stable, according as s is less than, equal to, or greater than $\cos^4 \alpha$.

409. *To find the metacentre of any solid of revolution when floating with its axis vertical.*

Let V be the magnitude of the part immersed, and a the radius of the plane of floatation.

Let A be the magnitude of the parts raised and depressed by the disturbance, and p the distance of their centres of gravity from C; then, as in Art. 407,

$$GO = gG - \frac{2Ap}{V\theta}.$$

The parts raised and depressed may be regarded as parts of a sphere (radius $= a$,) cut out by planes intersecting at the centre at the indefinitely small angle θ; and therefore (Art. 325),

$$p = \frac{3\pi a}{16}.$$

Also, $$A = \frac{2a^3\theta}{3}.$$

Substituting these values, we have

$$GO = gG - \tfrac{1}{4} \cdot \frac{\pi a^4}{V}.$$

Cor. The second term on the right hand side of the equation is the distance of the metacentre from the centre of gravity of the displaced fluid, or

$$gO = \tfrac{1}{4} \cdot \frac{\pi a^4}{V}.$$

410. Ex. 1. To find the metacentre of a right cone when floating with its axis vertical.

Let h be the height of the cone, 2α its vertical angle, and s its specific gravity relatively to the fluid. Then, if h' be the depth of

the part immersed, the radius of the plane of floatation is equal to

$$h' \tan a,$$

and the volume of the part immersed is equal to

$$\tfrac{1}{3}\pi h'^3 \tan^2 a.$$

Also, the distance between the centres of gravity of the cone and of the part immersed is equal to

$$\tfrac{3}{4}(h - h').$$

Substituting these values in the general formula

$$GO = gG - \tfrac{1}{4}\frac{\pi a^4}{V},$$

we have for the metacentre of a right cone,

$$GO = \tfrac{3}{4}(h - h') - \tfrac{3}{4}h' \tan^2 a,$$

$$= \tfrac{3}{4}\left(h - \frac{h'}{\cos^2 a}\right).$$

But, by Art. 402, $h' = h\sqrt[3]{s}$; therefore,

$$GO = \frac{3h}{4}\left(1 - \frac{\sqrt[3]{s}}{\cos^2 a}\right).$$

COR. Hence, the equilibrium of a right cone is stable when

$$\cos^2 a < \sqrt[3]{s};$$

$\therefore$ when $\quad \cos^6 a < s.$

411. Ex. 2. To find the metacentre of a circular cylinder when floating with its axis vertical.

Let r be the radius of the cylinder, h its height, and s its specific gravity relatively to that of the fluid; then, if h' be the depth of the part immersed,

$$V = \pi r^2 h';$$

also, $\quad a = r,$

and $\quad gG = \tfrac{1}{2}(h - h').$

Substituting these values in the general expression of Art. 409, we have

$$GO = \tfrac{1}{2}(h - h') - \tfrac{1}{4}\frac{r^2}{h'}.$$

But $h' = hs$, and therefore

$$GO = \frac{2h^2 s(1 - s) - r^2}{4hs}.$$

COR. Hence, the equilibrium is stable when

$$r^2 > 2h^2 s(1 - s);$$

that is, when
$$\frac{r^2}{h^2} > 2s(1 - s).$$

412. Ex. 3. To find the metacentre of a paraboloid of revolution.

Let $y^2 = 4cx$ be the equation of the generating parabola, h the axis of the paraboloid, and h' the depth of the part immersed; then

$$a^2 = 4ch',$$

and
$$V = 2\pi ch'^2.$$

Also (Art. 322),
$$gG = \tfrac{2}{3}(h - h');$$

therefore
$$GO = \tfrac{2}{3}(h - h') - 2c.$$

To find h', let w be the weight of a unit of the fluid; then, since the weight of the body is equal to that of the displaced fluid,

$$2\pi ch^2 . ws = 2\pi ch'^2 w;$$

whence
$$h' = h\sqrt{s}.$$

Consequently, for the metacentre of the paraboloid, we have

$$GO = \tfrac{2}{3}h(1 - \sqrt{s}) - 2c.$$

CHAPTER XI.

ON THE ATMOSPHERE AND THE MEASUREMENT OF HEIGHTS BY THE BAROMETER.

413. *To find the height of the atmosphere when supposed to be throughout of the same density as at the surface of the earth.*

Let h be the height of the barometric column, and w the weight of a cubic unit of the mercury. Let h' be the required height of the atmosphere, and w' the weight of a cubic unit of air of the mean density at the surface of the earth; then

$$w'h' = wh,$$

or,
$$h' = h\frac{w}{w'}.$$

But if s and s' be the specific gravities respectively of mercury and air,

$$\frac{w}{w'} = \frac{s}{s'};$$

$$\therefore \quad h' = h\frac{s}{s'}.$$

The mean value of h is 30 inches, $s = 13{\cdot}58$, and $s' = {\cdot}0012277$;

$$\therefore \quad h' = \frac{30 \times 13{\cdot}58}{{\cdot}0012277},$$

$$= \frac{5 \times 6{\cdot}79}{{\cdot}0012277} \text{ feet},$$

$$= 5{\cdot}18 \text{ miles}.$$

414. Common observation teaches that the atmosphere is not of uniform density, but the air being a compressible fluid, its density decreases as we rise from the surface of the earth.

The atmosphere cannot however, at least in the sense of a mass of gaseous matter revolving with the earth, be of unlimited extent. For each particle is acted upon by the centrifugal force, tending to draw it away from the centre, and by the force of gravitation tending to draw it towards the centre. As the distance from the earth increases, the centrifugal force increases, but the force of gravitation diminishes. There must, consequently, be a point at which these two forces are in equilibrium; and beyond this, the centrifugal force being greater than the force of gravitation, will cause all particles to fly off.

415. *To find the distance from the earth at which the centrifugal force is equal to the force of gravity.*

Let F be the centrifugal force for a particle on the surface of the earth at the equator, and F′ that for a particle at any height z above the surface. Since the time of revolution of each particle is the same, the centrifugal forces will be as the circumferences described, and therefore as the radii.

Let $r =$ the radius of the earth, then

$$F' : F :: r + z : r;$$

$$\therefore \qquad F' = F\frac{r+z}{r}.$$

But if G be the force of gravity at the equator (or 32·125), then, by calculating F from the known values of the earth's equatorial radius and the time of its rotation, it will be found that

$$F = \frac{G}{289};$$

$$\therefore \qquad F' = G\frac{r+z}{289 \cdot r}.$$

But if G′ be the force of gravity at the height z,

$$G' : G :: \frac{1}{(r+z)^2} : \frac{1}{r^2};$$

$$\therefore \qquad G' = G \cdot \left(\frac{r}{r+z}\right)^2.$$

But F′ and G′ must be equal. Therefore,

$$\frac{r+z}{r} = \sqrt[3]{289};$$

$$\therefore \qquad z = r\,(\sqrt[3]{289} - 1) = 5{\cdot}61 \,.\, r.$$

Or the greatest possible height to which the atmosphere can extend is about five-and-a-half times the earth's radius; that is, about 22,000 miles.

416. *To find the height of a station by means of a barometer, when the temperature and force of gravity are supposed to be the same at all altitudes.*

Let z be the height of the station, and let the atmosphere between the station and the earth be supposed to be divided into an indefinite number n of strata of equal thickness. Let τ be the thickness of each strata, then $n\tau = z$. Let ρ be the density of the atmosphere at the surface of the earth, and ρ_1, ρ_2, ρ_3, &c. the density at each of the successive levels. Let p, p_1, p_2, &c. in like manner denote the atmospheric pressures at the surface and at the different levels.

The difference between p and p_1 is the weight of a column of the lowest stratum, having a square unit for its base; and, since the stratum is indefinitely thin, its density throughout may be

regarded as uniform, and equal to that at the lowest point. Therefore,

$$p - p_1 = g\rho\tau.$$

But (Art. 248) $p = k\rho\,(1 + a\theta)$, and $p_1 = k\rho_1\,(1 + a\theta)$; therefore

$$\rho - \rho_1 = \frac{g\rho\tau}{k\,(1 + a\theta)};$$

$$\therefore \qquad \frac{\rho_1}{\rho} = 1 - \frac{g\tau}{k\,(1 + a\theta)}.$$

Let c stand for the fraction $\frac{g}{k\,(1 + a\theta)}$, then

$$\frac{\rho_1}{\rho} = 1 - c\tau.$$

Similarly,

$$\frac{\rho_2}{\rho_1} = 1 - c\tau,$$

$$\frac{\rho_3}{\rho_2} = 1 - c\tau,$$

&c. &c.

$$\frac{\rho_n}{\rho_{n-1}} = 1 - c\tau.$$

Multiplying these expressions together,

$$\frac{\rho_n}{\rho} = (1 - c\tau)^n = 1 - nc\tau + \frac{n\,.\,(n-1)}{2}c^2\tau^2 - \frac{n\,.\,(n-1)\,.\,(n-2)}{2\,.\,3}c^3\tau^3 + \&c.$$

$$= 1 - cz + \frac{z\,.\,(z-\tau)}{2}c^2 - \frac{z\,.\,(z-\tau)\,.\,(z-2\tau)}{2\,.\,3}c^3 + \&c.$$

But the more τ diminishes, the more is the error arising from the supposition that the density is uniform throughout the strata

diminished also. As τ diminishes, the more does the right-hand side of the above equation approach to

$$1 - cz + \frac{c^2z^2}{2} - \frac{c^3z^3}{2.3} + \&c.$$

or, to ϵ^{-cz}. Hence, ultimately,

$$\frac{\rho_n}{\rho} = \epsilon^{-cz};$$

$$\therefore \qquad \log\frac{\rho_n}{\rho} = -cz = -\frac{gz}{k(1+a\theta)}.$$

Let h' be the height of the barometric column at the station, and h its height at the earth's surface;

$$\text{then} \qquad \frac{h'}{h} = \frac{p_n}{p} = \frac{\rho_n}{\rho}, \qquad \frac{h'}{h} = \frac{h_n}{h} = \frac{\rho_n}{\rho};$$

$$\therefore \qquad \log\frac{h'}{h} = -\frac{gz}{k(1+a\theta)},$$

$$\text{or,} \qquad \log_\epsilon\frac{h}{h'} = \frac{gz}{k(1+a\theta)}.$$

417. Hence, if h_1 be the height of the barometric column at a station whose altitude is z_1 and h_2 that at another station whose altitude is z_2,

$$\log\frac{h}{h_1} = \frac{gz_1}{k(1+a\theta)},$$

$$\text{and} \qquad \log\frac{h}{h_2} = \frac{gz_2}{k(1+a\theta)};$$

$$\therefore \qquad \log\frac{h_2}{h_1} = \frac{g(z_1-z_2)}{k(1+a\theta)},$$

$$\text{or,} \qquad z_1 - z_2 = \frac{k(1+a\theta)}{g}\{\log_\epsilon h_2 - \log_\epsilon h_1\}.$$

418. The formula deduced in Art. 416 gives an approximation only to the altitude of any station, since the conditions under which alone it is strictly applicable are in no case fulfilled; for the temperature of the atmosphere is not uniformly diffused through all altitudes, nor does the force of gravity remain the same. The law of the diffusion of temperature is not known, and the formula cannot be corrected for this cause of variation; the temperature, therefore, when the stations are not greatly distant, is commonly treated as uniform, and taken as the mean of the temperatures at the two stations. The expression in Art. 416 as corrected, for the variation in the force of gravity, is

$$\log\frac{h}{h'} = \frac{grz}{k(1+a\theta)(r+z)},$$

where $r =$ the radius of the earth.*

419. In order to apply the preceding formula to the practical determination of heights, it is necessary to calculate the value of the constant k. By Art. 248, $k = \frac{p}{\rho}$ when ρ is the density of the atmosphere at the freezing point, under a standard pressure p. When the barometer stands at 30 inches, the density of atmospheric air at the freezing point is found to bear to that of mercury the ratio of 1 : 10462. Let s be the density of mercury, and g the force of gravity, then

$$p = 30gs,$$

and

$$\rho = \frac{s}{10462};$$

$$\therefore \quad \frac{k}{g} = 313860 \text{ inches},$$

$$= 8718{\cdot}33 \text{ yards}.$$

Substituting this in the formula of Art. 417,

$$z_1 - z_2 = 8718{\cdot}33\,(1 + a\theta)\,(\log h_2 - \log h_1).$$

* It must be carefully noted, that the logarithms employed in Arts. 416, 417, and 418 are Napierian logarithms.

Let τ_1 and τ_2 be the temperatures at the two stations, expressed in degrees above the freezing point. Substituting $\frac{1}{2}(\tau_1 + \tau_2)$ for θ, $\frac{1}{490}$ for a, and using common instead of Napierian logarithms, we obtain

$$z_1 - z_2 = \frac{8718{\cdot}33}{M}\left(1 + \frac{\tau_1 + \tau_2}{980}\right)(\log h_2 - \log h_1);$$

where M is the modulus of the common system of logarithms, or ·43429. Hence,

$$z_1 - z_2 = 20075\left(1 + \frac{\tau_1 + \tau_2}{980}\right)(\log h_2 - \log h_1).$$

The pressure of vapour in the atmosphere renders some slight change necessary in the numbers here given, and values somewhat higher than the above are in consequence usually given to k and a. These are

$$\frac{k}{g} = 20117 \text{ yards, and } a = \frac{1}{450};$$

whence $\quad z_1 - z_2 = 20117\left(1 + \frac{\tau_1 + \tau_2}{900}\right)(\log h_2 - \log h_1).$

Neglecting the correction for temp. and remembering 1 fath. = 2 yds we have fath. $(z_1 - z_2) = 10{,}000(\log h_2 - \log h_1)$

420. *To correct the barometric column for the rise or fall of the mercury in the cistern.*

When the mercury rises in the tube of a barometer, the variation in the height of the column, as denoted by the accompanying scale, does not express the entire amount of variation; since, while the mercury has risen in the tube, it has fallen in the cistern, and the true variation in the barometric column is the observed variation + the depth the mercury has fallen in the cistern. In like manner, when the mercury falls in the tube it rises in the cistern, and the observed variation is greater than the real variation by the height over which the mercury has risen in the cistern.

Let h be the observed variation in the barometric column, and x the depth the mercury has risen or fallen in the cistern. Let A be the area of a horizontal section of the cistern, a_1 that of the

tube, a_2 that of its bore. The quantity of mercury that has been gained or lost by the tube is ha_2, and that which has been gained or lost by the cistern is $(A - a_1)\,x$; therefore,

$$(A - a_1)\,x = ha_2,$$

or,

$$x = \frac{a_2}{A - a_1} \cdot h.$$

421. *To correct the barometric column for the change of temperature.*

As the temperature changes, the density of the mercury changes, and consequently the column which denotes a given atmospheric pressure will vary as the temperature varies. A correction therefore must be made in the observed height of the column for the expansion or contraction of the mercury resulting from the change of temperature. It is found that, for each degree Fahrenheit, mercury expands $\frac{1}{10000}$ of its bulk at the freezing point. Hence, if τ be the temperature of the mercury expressed in degrees above the freezing point, h the observed height of the barometric column, and h' the corrected height for the standard temperature,

$$h = h'\left(1 + \frac{\tau}{10000}\right);$$

$$\therefore \qquad h' = h\left(1 + \frac{\tau}{10000}\right)^{-1}$$

$$= h\left(1 - \frac{\tau}{10000}\right) \text{ nearly};$$

whence, to correct the column for the temperature from the observed height, subtract its ten-thousandth part as many times as there are degrees of temperature above the freezing point.

MISCELLANEOUS EXAMPLES.

1. The plane surface of a spherical segment is the base of a cone; what must be the height of the cone that the solid may rest with any point of its spherical surface in contact with a horizontal plane, the axis of the segment being equal to half its radius?

Let r be the radius of the segment, then the height of the cone is equal to

$$\frac{r(4+\sqrt{52})}{4}.$$

2. A cylindrical vessel, filled with liquid, is divided into four equal portions by two vertical planes through its axis at right angles to each other; find the resultant pressure upon either portion of the cylindrical surface.

Let a be the height of the vessel, r the radius of its base, and w the weight of a unit of the liquid, then the required pressure is equal to

$$\frac{wa^2r\sqrt{2}}{2}.$$

3. Two balls of equal weight are fastened together by a cord of given length, one of the balls is placed on the ground, and the other projected vertically upwards with a given velocity, to what height will the second ball rise?

Let l be the length of the cord, and the velocity of projection that acquired by a body falling freely through the height h, then the required distance is equal to

$$\frac{h-l}{4}.$$

4. Find, by Guldin's Properties, the surface of the spindle generated by the revolution of a circular arc about its chord.

Let r be the radius of the arc, and 2α the angle subtended by the arc at the centre of the circle, then the surface of the spindle is equal to

$$4\pi r^2 (\sin \alpha - \alpha . \cos \alpha).$$

5. A hemispherical bowl, filled with a fluid, is divided into two parts by a vertical plane through the centre, find the resultant pressure upon one of the parts.

Let r be the radius of the bowl, and w the weight of a unit of the fluid, then the pressure required is equal to

$$\tfrac{1}{3} w r^3 \sqrt{(\pi^2 + 4)}.$$

6. A body is projected with a given velocity along a rough horizontal plane; find the space described by the body before coming to rest.

Let μ be the co-efficient of friction, and the velocity of projection that acquired in falling freely through the distance h, then the space required is equal to

$$\frac{h}{\mu}.$$

7. A body is projected in a horizontal direction along a smooth inclined plane with a velocity equal to that acquired in falling freely through the distance h; show that the path described is a parabola, and that if α be the inclination of the plane, the latus rectum of the parabola is equal to

$$\frac{4h}{\sin \alpha}.$$

8. Find the velocity acquired by a body in falling down a rough inclined plane.

Let v_1 v_2 be the velocities acquired by a body in falling freely through distances equal to the height and base of the plane

respectively, and let μ be the co-efficient of friction, then the required velocity is equal to

$$\sqrt{(v_1^2 - \mu v_2^2)}.$$

9. An imperfectly elastic ball is projected at a given inclination against a horizontal plane; determine the velocity of projection that the ball after its rebound may strike a given point in a vertical wall.

Let e be the modulus of elasticity, α the angle at which the ball strikes the horizontal plane, a the distance of the wall from the first point of impact, and b the height of the second point, then the velocity of projection is that acquired in falling freely through a distance equal to

$$\frac{a^2}{2ae \sin 2\alpha - 4b \cos^2\alpha}.$$

10. In a cylindrical vessel filled with air, a cone exactly fitting it is placed with its vertex downwards, to what depth will the cone sink, supposing it to come to rest before its vertex touches the bottom of the vessel.

Let a be the height of the cylinder, $3b$ that of the cone, s the specific gravity of the cone relatively to that of mercury, and h the height of the barometric column; then the required depth is equal to

$$\frac{bs(a-b)}{h+bs}.$$

11. A lever of equal arms, bearing equal weights P, P, is moveable about a rough cylindrical axis; find how far one of the weights may be moved towards the fulcrum without disturbing the equilibrium, disregarding the difference between the radius of the axis and that of the hole in which it works.

Let w be the weight of the lever, r the radius of the axis, and μ the co-efficient of friction, then, if $\tan \epsilon = \mu$, the distance required is equal to

$$\frac{r \sin \epsilon \, (2P + w)}{P}.$$

12. Determine the velocity with which a body must be projected up a rough inclined plane so that it may just reach the top, the inclination of the plane being equal to the angle of repose.

Let h be the height of the plane, then the required velocity is equal to

$$\sqrt{(4gh)}.$$

13. A thin conical vessel, when floating in a certain fluid with its vertex downwards, is immersed to half its depth, determine the nature of the equilibrium.

Let h be the height of the cone, then the equilibrium is stable if the radius of the base is greater than

$$\frac{h\sqrt{7}}{3}.$$

14. A body falls down an inclined plane which is partly smooth and partly rough, what must be the inclination of the plane, that the velocity acquired in falling down a given distance on the smooth plane shall be just sufficient to carry the body over an equal distance along the rough plane?

Let μ be the co-efficient of friction, then the inclination of the plane is equal to

$$\tan^{-1}\left(\frac{\mu}{2}\right).$$

15. Two planes of equal inclination meet in their lowest points, a perfectly elastic ball falling vertically strikes one of the planes at a given point, and, after rebounding from the second plane, rises vertically, from what height did the ball fall on to the first plane?

Let α be the inclination of the planes, and a the distance of the first point of impact from the lowest point of the plane, then the height required is equal to

$$\frac{a \cos \alpha}{\sin 4\alpha}.$$

16. A perfectly elastic ball falls into a hemispherical bowl from a given height, determine the position of the first point of impact, in order that the second may be the lowest point of the bowl.

Let h be the given height, and r the radius of the bowl, then, if θ be the angle subtended at the centre by the arc contained by the two points of impact,

$$\cos\theta = \frac{1}{4} + \frac{1}{4}\sqrt{\left(1 + \frac{r}{h}\right)}.$$

17. A circular plate just immersed in a liquid is divided by a horizontal diameter; find the centres of pressure of the upper and lower semicircles.

Let r be the radius of the circle, then the distance of the centre of pressure of the upper semicircle from the highest point of the plate is equal to

$$\frac{r}{4} \cdot \frac{15\pi - 32}{3\pi - 4},$$

and that of the lower semicircle is equal to

$$\frac{r}{4} \cdot \frac{15\pi + 32}{3\pi + 4}.$$

18. An elastic ball is projected with a given velocity from a point in a horizontal plane; at what distance from that point will it finally rest?

Let e be the modulus of elasticity, $\sqrt{(2gh)}$ the velocity, and a the angle of projection, then the distance required is equal to

$$\frac{2h \sin 2a}{1 - e}.$$

19. The axis of a thin hemispherical bowl, when on the point of slipping down an inclined plane, is inclined to the vertical line at an angle of $45°$; determine the co-efficient of friction.

The co-efficient of friction is equal to

$$\frac{2\sqrt{2} + 1}{7}.$$

20. An elastic ball is projected from a point in one of two parallel vertical walls, and strikes against the other; compare the intervals between the times of the successive impacts.

Let t be the time between projection and the first impact, then the required intervals are

$$\frac{t}{e}, \ \frac{t}{e^2}, \ \frac{t}{e^3}, \text{ and so on.}$$

21. A hemispherical bowl, of given material, just floats in a given fluid; determine the thickness of the bowl.

Let s be the specific gravity of the material of the bowl relatively to the given fluid, then, if a be the external radius, the internal radius is equal to

$$a\sqrt{\left(\frac{s-1}{s}\right)}.$$

22. A particle moves from rest over the convex side of a parabola whose axis is vertical; show that when the particle starts from the vertex, the square of the reaction at any point varies inversely as the cube of the distance from the focus.

23. A particle moves from rest over the convex side of a parabola, whose axis is horizontal and whose plane is vertical; show that when the particle starts from a point whose distance from the axis is equal to three times the semi latus rectum, it will leave the curve at the extremity of the latus rectum.

APPENDIX.

I. *To find the length of a perpendicular* (p) *drawn to the tangent of an ellipse from any point* (S).

Let $a^2y^2 + b^2x^2 = a^2b^2$ be the equation to the ellipse, the centre being the origin of co-ordinates, then the equation to the tangent at the point (x', y') is

$$a^2yy' + b^2xx' = a^2b^2;$$

and $\therefore$ if x, y be the co-ordinates of S,

$$p = \pm\frac{a^2yy' + b^2xx' - a^2b^2}{\sqrt{(a^4y'^2 + b^4x'^2)}};$$

$$= \pm\frac{a^2yy' + b^2xx' - a^2b^2}{ab\sqrt{(rr')}}.$$

COR. 1. If S be at the centre, $x = 0$, and $y = 0$,

$$\therefore \qquad p = \frac{ab}{\sqrt{(rr')}}.$$

COR. 2. If S be at the focus from which r is measured, $x = -ae$, and $y = 0$,

$$\therefore \qquad p = b\sqrt{\left(\frac{r}{r'}\right)}.$$

COR. 3. If S be at the extremity of the major axis, or $x = -a$, and $y = 0$,

$$\therefore \qquad p = \frac{b(x'+a)}{\sqrt{(rr')}} = \frac{b \, . \, SM}{\sqrt{(rr')}}.$$

COR. 4. If S be at any point on the ellipse, then $a^2y'^2 + b^2x'^2 = a^2b^2$, and $a^2y^2 + b^2x^2 = a^2b^2$,

$$\therefore \qquad 2a^2b^2 = a^2(y^2 + y'^2) + b^2(x^2 + x'^2)$$

But $$p = -\frac{2a^2yy' + 2b^2xx' - 2a^2b^2}{2ab\sqrt{(rr')}},$$

the negative sign being taken because S is on the same side of the tangent as the origin.

Substituting for $2a^2b^2$, the value given above, we have

$$p = \frac{a^2(y-y')^2 + b^2(x-x')^2}{2ab\sqrt{(rr')}}.$$

II. To find the length of a perpendicular (p) drawn to the tangent of a parabola from any point S.

Let $y^2 = 4cx$ be the equation to the parabola, the vertex being the origin; then the equation to tangent at (x', y') is

$$yy' - 2cx = 2cx';$$

and therefore, if x, y be the co-ordinates of S,

$$p = \pm \frac{yy' - 2cx - 2cx'}{\sqrt{(4c^2 + y'^2)}},$$

$$= \pm \frac{yy' - 2cx - 2cx'}{2\sqrt{(cr)}},$$

COR. 1. If S be at the focus, $x = c$ and $y = 0$,

$$\therefore \qquad p = \sqrt{(cr)},$$

COR. 2. If S be at the vertex, $x = 0$ and $y = 0$,

$$\therefore \qquad p = x'\sqrt{\left(\frac{c}{r}\right)}.$$

COR. 3. If S be any point on the parabola, then $y^2 = 4cx$, and

$$p = \frac{(y - y')^2}{4\sqrt{(cr)}}.$$

III. To find the length of perpendicular (p) drawn to the tangent of a hyperbola from any point (S).

Let $a^2y^2 - b^2x^2 = -a^2b^2$ be the equation to the hyperbola, then the equation to the tangent at (x', y') is

$$a^2yy' - b^2xx' = -a^2b^2,$$

and $\therefore$ if x, y be the co-ordinates of S,

$$p = \pm\frac{a^2yy' - b^2xx' + a^2b^2}{\sqrt{(a^4y'^2 + b^4x'^2)}}.$$

$$= \pm\frac{a^2yy' - b^2xx' + a^2b^2}{ab\sqrt{(rr')}}.$$

COR. 1. If S be at the centre, $x = 0$, and $y = 0$, and hence

$$p = \frac{ab}{\sqrt{(rr')}}.$$

COR. 2. If S be at the focus from which r is measured, $x = -ae$, and $y = 0$. Hence,

$$p = b\sqrt{\left(\frac{r}{r'}\right)}.$$

COR. 3. If S be any point in the hyperbola, then, proceeding as in I. Cor. 4, we have

$$p = \pm\frac{a^2(y - y')^2 - b^2(x - x')^2}{2ab\sqrt{(rr')}},$$

the upper sign if S and P are on the same branch, the lower sign if they are on different branches.

RADIUS OF CURVATURE.

Def. If P and Q be any two points in a curve, and the circle be drawn which passes through P and Q, and has its centre in the normal at P, then the limiting value of this circle as Q approaches more and more nearly to P, is called the *circle of curvature*, and its radius the *radius of curvature*.

Let ρ be the radius of curvature at any point P (fig. Art. 177), then ρ is the limiting value of PC, the radius of the circle PQq. But

$$PQ^2 = 2PC \,.\, PX,$$

$$= 2PC \,.\, QR,$$

$$\therefore \qquad PC = \frac{PQ^2}{2QR},$$

$$\therefore \qquad \rho = \text{limit}\,\frac{PQ^2}{2QR}.$$

IV. *To find the radius of curvature at any point* (P) *of an ellipse.*

Let x, y be the co-ordinates of P, x', y', those of Q; then

$$PQ^2 = (x - x')^2 + (y - y')^2;$$

and since QR is the perpendicular from Q to the tangent at P, then (I. Cor. 4),

$$2QR = \frac{b^2(x - x')^2 + a^2(y - y'^2)}{ab\sqrt{(rr')}},$$

$$\therefore$$

$$\frac{PQ^2}{2 \,.\, QR} = ab\sqrt{(rr')} \,.\, \frac{(x - x')^2 + (y - y')^2}{b^2(x - x')^2 + a^2(y - y')^2},$$

$$= ab\sqrt{(rr')} \,.\, \frac{1 + \left(\dfrac{y - y'}{x - x'}\right)^2}{b^2 + a^2\left(\dfrac{y - y}{x - x'}\right)^2},$$

But the limit of $\frac{y-y'}{x-x'}$, as found in deducing the equation to the tangent, is $-\frac{b^2}{a^2}\frac{x}{y}$, $\therefore$

$$\rho = ab\sqrt{(rr')} \cdot \frac{1 + \frac{b^4x^2}{a^4y^2}}{b^2 + \frac{a^2b^4x^2}{a^4y^2}},$$

$$= ab\sqrt{(rr')} \cdot \frac{a^4y^2 + b^4x^2}{a^2b^2(a^2y^2 + b^2x^2)},$$

$$= ab\sqrt{(rr')} \cdot \frac{a^2b^2rr'}{a^4b^4},$$

$$= \frac{(rr')^{\frac{3}{2}}}{ab}.$$

V. To find the radius of curvature at any point of a parabola.

As in the preceding,

$$PQ^2 = (x-x')^2 + (y-y')^2,$$

and by II. Cor. 3,

$$2QR = \frac{(y-y')^2}{2\sqrt{(cr)}};$$

$$\therefore \quad \frac{PQ^2}{2QR} = 2\sqrt{(cr)} \cdot \frac{(x-x')^2 + (y-y')^2}{(y-y')^2};$$

$$= 2\sqrt{(cr)}\left\{\left(\frac{x-x'}{y-y'}\right)^2 + 1\right\}.$$

But the limit of $\frac{x-x'}{y-y'} = \frac{y}{2c}$; hence

$$\rho = 2\sqrt{(cr)}\left\{\left(\frac{y}{2c}\right)^2 + 1\right\},$$

$$= 2\sqrt{(cr)} \cdot \frac{4cx + 4c^2}{4c^2},$$

$$= 2\sqrt{(cr)} \cdot \frac{c+x}{c},$$

$$= 2c \cdot \left(\frac{r}{c}\right)^{\frac{3}{2}}.$$

VI. To find the radius of curvature at any point of a hyperbola.

As before,

$$PQ^2 = (x - x')^2 + (y - y')^2,$$

and by III. Cor. 3, since P and Q are on the same branch,

$$2QR = -\frac{b^2(x - x')^2 - a^2(y - y')^2}{ab\sqrt{(rr')}},$$

$$\therefore \qquad \frac{PQ^2}{2QR} = -ab\sqrt{(rr')} \cdot \frac{(x - x')^2 + (y - y')^2}{b^2(x - x')^2 - a^2(y - y')}.$$

$$= -ab\sqrt{(rr')} \cdot \frac{1 + \left(\dfrac{y - y'}{x - x'}\right)^2}{b^2 - a^2\left(\dfrac{y - y'}{x - x'}\right)^2},$$

But limit of $\dfrac{y - y'}{x - x'} = \dfrac{b^2x}{a^2y}$,

$$\therefore \qquad \rho = -ab\sqrt{(rr')} \; \frac{1 + \dfrac{b^4x^2}{a^4y^2}}{b^2 - \dfrac{a^2b^4x^2}{a^4y^2}},$$

$$= -ab\sqrt{(rr')} \; \frac{a^4y^2 + b^4x^2}{a^2b^2(a^2y^2 - b^2x^2)},$$

$$= ab\sqrt{(rr')} \cdot \frac{-a^2b^2(rr')}{-a^4b^4},$$

$$= \frac{(rr')^{\frac{3}{2}}}{ab}.$$

W. Brendon and Son, Printers, Plymouth.

Dr. W. Smith's Smaller School Books.

Fcp. 8vo., cloth, red edges. 3*s*. 6*d*. each.

A SMALLER HISTORY OF ENGLAND, for the use of Schools. Edited by Dr. WILLIAM SMITH. 68 Illustrations. Eighth Edition. Fcp. 8vo. 3s. 6d.

"This little volume (History of England) is so pregnant with valuable information, that it will enable any one who reads it attentively to answer such questions as are set forth in the English History Papers in the Indian Civil Service Examinations."—*Reader.*

A SMALLER HISTORY OF ROME, for the use of Schools. By Dr. WILLIAM SMITH. 79 Illustrations. Eighth Edition. Fcp. 8vo. 3s. 6d., cloth.

A SMALLER HISTORY OF GREECE, for the use of Schools. By Dr. WILLIAM SMITH. 74 Illustrations. Eighth Edition. Fcp. 8vo. 3s. 6d.

A SMALLER CLASSICAL MYTHOLOGY for the use of Schools. With extracts from the Poets in English, and Questions upon the Work. Edited by Dr. W. SMITH. Many Wood Cuts. Fcp. 8vo. 3*s*. 6*d*.

History of Rome, for Colleges and Schools.

DR. SCHMITZ'S HISTORY OF ROME, from the Earliest Times to the Death of Commodus. Twenty-third Thousand. 100 Illustrations. Small 8vo. 7s. 6d.

"The most prominent feature in the work is, as it appears to us, the great labour bestowed on the earlier portion, the development of the constitution, and the struggle between the two estates. While the difficulties of this subject are fairly grappled with, the discussion is not overladen with a needless display of abstruse learning, useless, and therefore mischievous, in a work designed for Schools. To the complete success of this effort to render intelligible to boys this most important, if not very attractive part of Roman history, we can testify from our own experience."—*Educational Times.*

Latin & Greek Authors. English Notes.

TACITUS, GERMANIA, AGRICOLA, AND First Book of the Annals. With English Notes. Edited by Dr. WILLIAM SMITH. Sixth Edition. 12mo. 5s.

By the same Editor,

PLATO. THE APOLOGY OF SOCRATES, the Crito and Part of the Phædo; with Notes in English from Stallbaum. Schleiermacher's Introductions. Fifth Edition. 12mo. 5s.